Manifolds associated with Bier spheres and generalized permutahedra

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Simplicial complexes

Let us fix our terminology and notations related to simplicial complexes.

Abstract simplicial complex

By an (abstract) simplicial complex K on $[m] := \{1, 2, \dots, m\}$ we mean a subset of $2^{[m]}$ s.t.

$$\sigma \in K, \tau \subseteq \sigma \Rightarrow \tau \in K.$$

If $\{i\} \in K$, then i is a vertex of K, otherwise i is a ghost vertex of K; elements of K are called its faces (or, simplices). The dimension of K is one less than the maximal number of elements in a face of K.

Minimal non-faces and maximal faces

The set of maximal faces (w.r.t. inclusion) of K will be denoted by $\mathrm{M}(K)$. A subset $I \subseteq [m]$ s.t. $K_I := K \cap 2^I = \partial\Delta_I$ is called a minimal non-face of K and we write: $I \in \mathrm{MF}(K)$.

$\mathcal{Z}_K(X,A) = (X,A)^K$

Let (X,A) be a pair of spaces and let K be an abstract simplicial complex, $K \subseteq 2^{[m]} \quad (m = N+1)$.

The associated *polyhedral product* (K-power, generalized moment-angle complex) is the space,

$$\mathcal{Z}_K(X,A) = \mathrm{colim}_{\sigma\in K}(X,A)^\sigma = \mathrm{colim}_{\sigma\in K}(\prod_{i\in\sigma} X \times \prod_{j\notin\sigma} A) \subseteq X^m.$$

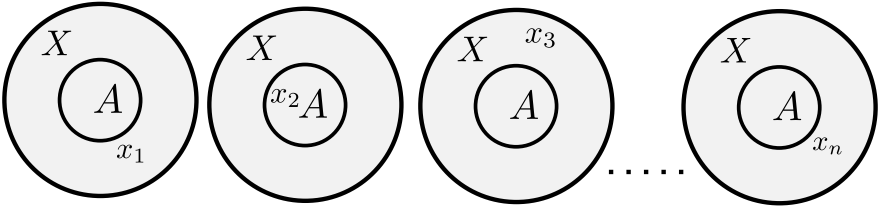
For $x = (x_i) \in X^m$ let $H_A(x) := \{i \in [m] \mid x_i \in A\}$ be the A-hitting set of x.

Then

$$\mathcal{Z}_K(X,A) = \{x \in X^m \mid H_A(x) \in W\}$$

where $W := 2^{[m]} \setminus K$.

$\mathcal{Z}_K(X, A) = (X, A)^K$



$$H_A(x) := \{i \in [n] \mid x_i \in A\} \quad M_A(x) = \{j \in [n] \mid x_i \notin A\}\,.$$

Definition: If $(X, A) := (D^2, S^1)$ we obtain the moment-angle complex

$$\mathcal{Z}_K := \mathcal{Z}_K(D^2, S^1) = (D^2, S^1)^K\,.$$

Combinatorial spheres and moment-angle-complexes

Let K be a simplicial complex on $[m]$ with $\dim K = n-1, n \geq 3$.

Theorem (Cai Li'17)

- $\mathcal{R}_K$ is a topological n-manifold $\Leftrightarrow$ K is a s.c. homology sphere;
- $\mathcal{Z}_K$ is a topological $(n+m)$-manifold $\Leftrightarrow$ K is a homology sphere.

Theorem (Panov, Ustinovsky'12; Tambour'12)

If K is a starshaped sphere, then $\mathcal{R}_K$ and $\mathcal{Z}_K$ are both homeomorphic to smooth manifolds.

Bier spheres: general properties II

It turns out that all Bier spheres satisfy the conditions of the last theorem.

Theorem (Jevtić, Timotijević, Živaljević'19)

Let $K \subset 2^{[m]}$ be a simplicial complex. Then $\mathrm{Bier}(K)$ has a geometric realization as a starshaped sphere in the hyperplane $H_0 := \{x \in \mathbb{R}^m \mid \langle u, x \rangle = 0\}$, where u is the sum of the standard basis vectors $e_i, 1 \leq i \leq m$ in $\mathbb{R}^m$.

Thus, (real) moment-angle-complexes over Bier spheres acquire equivariant smooth structures.

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Simple games

Simple games (von Neumann and Morgenstern 1944; Shapley 1962; Taylor and Zwicker 1999, etc.) model the distribution of power among coalitions of players.

- A *simple game* is a family $W \subseteq 2^P$ such that $P \in W, \emptyset \notin W$ and $A \supseteq B \in W \Rightarrow A \in W$.
- A simplicial complex $K \subseteq 2^V$ on a set of vertices V is a *down-set* $(A \subset B \in K \Rightarrow A \in K)$ such that $\emptyset \in K$.

$W \subseteq 2^P$ is a simple game $\Leftrightarrow 2^P \setminus W$ is a simplicial complex

Weighted majority games

An important class of simple games are the *weighted majority games*, where each player $i \in P$ is associated a *weight* $w_i \in \mathbb{R}^+$ and the *winning coalitions* are sets $A \subseteq P$ whose total weight is above a certain threshold q, prescribed in advance.

$$A \in \Gamma \Leftrightarrow w(A) \geqslant q\,.$$

The corresponding set of *losing coalitions* is the *threshold simplicial complexes* $Tr_{w<q}$.

Roughly weighted simple games

Definition. A simple game (P, W), where $K = 2^P \setminus W$ is the collection of losing coalitions, is called *roughly weighted* if there exist strictly positive real numbers $w = (w_1, \ldots, w_n)$ and a positive real number q (called the quota) such that for each $X \in 2^P$

$$w(X) = \sum_{i \in X} w_i > q \quad \Rightarrow \quad X \in W \tag{1}$$

$$w(X) = \sum_{i \in X} w_i < q \quad \Rightarrow \quad X \in K \tag{2}$$

$$w(X) = \sum_{i \in X} w_i = q \quad \Rightarrow \quad ??? \tag{3}$$

Trading transforms

The class of *roughly weighted simple games* is considerably larger then the class of weighted majority games. Recall that both classes can be characterized in terms of "trading transforms", by the results of Elgot (1961), and Taylor and Zwicker (1992), for the weighted majority games, and Gvozdeva and Slinko (2011), for the roughly weighted games.

$(X_1, \dots, X_k; Y_1, \dots, Y_k)$ is a trading transform if

- $X_i \in W, Y_i \in K$ for all $i \in [k]$,
- $\varphi_{X_1} + \cdots + \varphi_{X_k} = \varphi_{Y_1} + \cdots + \varphi_{Y_k}$

where φ_A is the characteristic (indicator) function of A.

Example

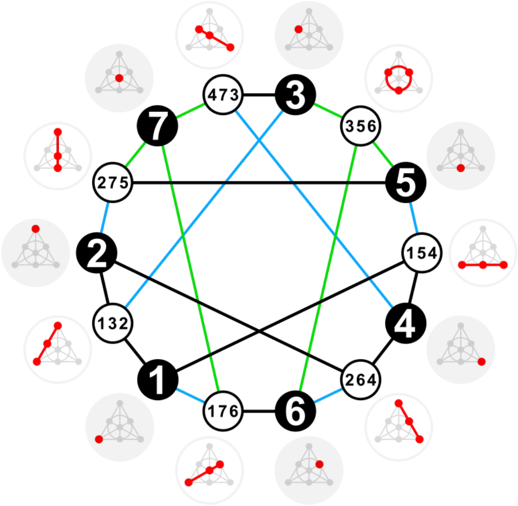
$$K = 2^{\{1,2\}} \cup 2^{\{2,3\}} \cup 2^{\{3,4\}} \cup 2^{\{4,1\}}$$

The trading transform

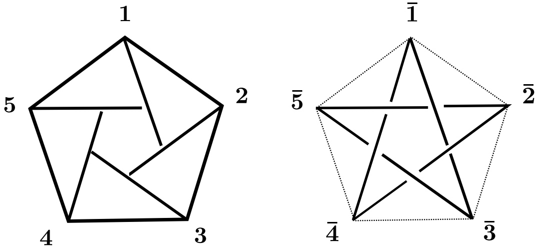
$$\varphi_{\{1,2\}} + \varphi_{\{2,3\}} + \varphi_{\{3,4\}} + \varphi_{\{4,1\}} = 2\varphi_{\{1,3\}} + 2\varphi_{\{2,4\}}$$

serves as a certificate of non weightedness showing that K is not a threshold complex.

Fano game



Möbius game



$$\varphi_{\{1,2,3\}} + \varphi_{\{2,3,4\}} + \varphi_{\{3,4,5\}} + \cdots = \varphi_{\{1,2,4\}} + \varphi_{\{2,3,5\}} + \varphi_{\{3,4,1\}} + \ldots$$

$$\varphi_{\{\bar{1},\bar{2}\}}+\varphi_{\{\bar{2},\bar{3}\}}+\varphi_{\{\bar{3},\bar{4}\}}+\varphi_{\{\bar{4},\bar{5}\}}+\varphi_{\{\bar{5},\bar{1}\}} = \varphi_{\{\bar{1},\bar{3}\}}+\varphi_{\{\bar{3},\bar{5}\}}+\varphi_{\{\bar{5},\bar{2}\}}+\varphi_{\{\bar{2},\bar{4}\}}+\varphi_{\{\bar{4},\bar{1}\}}$$

Bier sphere $B(K, K^\circ)$

$$K * L = \{A \uplus B \mid A \in K, B \in L\}\,.$$

$$K *_\Delta L = \{A \uplus B \mid A \in K, B \in L \text{ and } A \cap B = \emptyset\}\,.$$

$$K^\circ = \{A \subset [m] \mid A^c \notin K\} \qquad \text{is the Alexander dual of } K\,.$$

$$Bier(K) = B(K, K^\circ) := K *_\Delta K^\circ$$

is the associated *Bier sphere*.

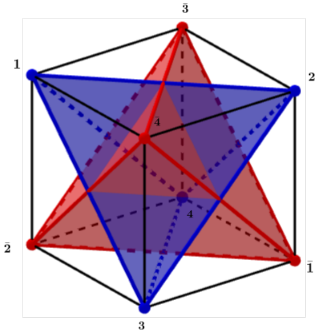
If $Vert(K) = [n] = \{1, 2, \dots n\}$, $Vert(K^\circ) = [\bar{n}] = \{\bar{1}, \bar{2}, \dots \bar{n}\}$ then $Vert(B(K, K^\circ)) = [n] \cup [\bar{n}]$ and $(A, B, C) \in B(K, K^\circ)$ is equivalent to

- $[n] = A \uplus B \uplus C$ (disjoint union);
- $A \in K$ and $\bar{C} := \{\bar{k}\}_{k \in C} \in K^\circ$;
- $\emptyset \neq B \neq [n]$.

Glossary

$Bier(K) = K *_\Delta K^\circ$, the Bier sphere of K, is a combinatorial object (simplicial complex), defined as a deleted join of two simplicial complexes (K and its Alexander dual K°).

$Fan(K) = BierFan(K)$, the *canonical* or the *Bier fan* of K, is a complete, simplicial fan in $H_0 \cong \mathbb{R}^{n-1}$, associated to a simplicial complex $K \subsetneq 2^{[n]}$.



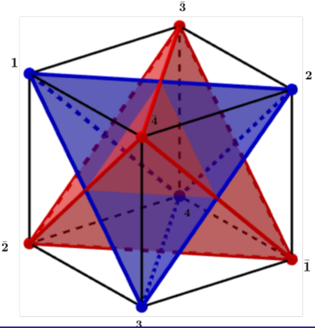
Example

$K = \Delta_{[4]}^{(1)}$ (1-skeleton of $\Delta_{[4]}$)

$K^\circ = \Delta_{[4]}^{(0)}$ (0-skeleton of $\Delta_{[4]}$)

$Bier(K) = K *_\Delta K^\circ$, the geometric realization of $Bier(K)$, is a triangulated boundary of the cube (diplo-simplex).

$Fan(K) = BierFan(K)$, the *canonical* or the *Bier fan* of K, is the radial fan of the triangulated cube.



Edmonds-Fulkerson bottleneck thm.

Bottleneck Extrema

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ABSTRACT

Let E be a finite set. Call a family of mutually noncomparable subsets of E a clutter on E. It is shown that for any clutter $\mathscr{R}$ on E, there exists a unique clutter $\mathscr{S}$ on E such that, for any function f from E to real numbers,

$$\min_{R \in \mathscr{R}} \max_{x \in R} f(x) = \max_{S \in \mathscr{S}} \min_{x \in S} f(x).$$

Specifically, $\mathscr{S}$ consists of the minimal subsets of E that have non-empty intersection with every member of $\mathscr{R}$. The pair $(\mathscr{R}, \mathscr{S})$ is called a blocking system on E. An algorithm is described and several examples of blockings systems are discussed.

Bottleneck theorem and discrete Morse theory

$$\min_{A\in\mathcal{A}} \max_{x\in A} f(x) = \max_{B\in\mathcal{B}} \min_{x\in B} f(x) = f(c) \tag{4}$$

Let $K := 2^{[n]} \setminus \mathcal{A}$ and $L = K^\circ := 2^{[n]} \setminus \mathcal{B}$ and let $Bier(K) = K *_\Delta K^\circ \cong S^{n-2}$ be the associated Bier sphere. Then $f : [n] \to \mathbb{R}$ (assumed to be 1–1) induces a perfect (discrete) Morse function on $Bier(K)$ with the critical cell in dimension $(n-2)$ of the form $(X, c, Y) \in Bier(K) = K *_\Delta K^\circ$.

D. Jojić, G. Panina, S. Vrećica, R. Živaljević. Generalized chessboard complexes and discrete Morse theory. Chebyshevskii Sbornik, 2020, Volume 21, Issue 2, 207–227.

Simplicial Steinitz problem and Bier spheres

The problem of deciding if a given triangulation of a sphere is realizable as the boundary sphere of a simplicial, convex polytope is known as the "Simplicial Steinitz problem"

G. Ewald: *Combinatorial Convexity and Algebraic Geometry*, volume 168 of Graduate Texts in Mathematics. Springer-Verlag, 1996.

Vast majority of *Bier spheres* $B(K, K^\circ)$ are "non-polytopal", in the sense that they are not combinatorially isomorphic to the boundary of a convex polytope.

Characterization of weighted majority simple games

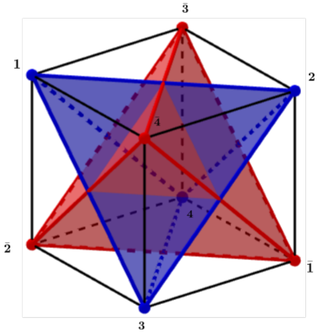
Theorem 1. ([3]) Suppose that $K \subsetneq 2^{[n]}$ is a proper simplicial complex such that $Vert(K) = [n]$. Then K is a threshold complex (equivalently $W = 2^{[n]} \setminus K$ is a weighted majority game with all weights strictly positive) if and only if the canonical fan $Fan(K)$ of K is polytopal.

Conclusion: Weighted majority games correspond to canonically polytopal Bier spheres!

$Fan(K) = BierFan(K)$

$Bier(K) = K *_\Delta K^\circ$, the geometric realization of $Bier(K)$, is a triangulated boundary of the diplo-simplex.

$Fan(K) = BierFan(K)$, the *canonical* or the *Bier fan* of K, is the radial fan of the diplo-simplex.



Characterization of roughly weighted simple games

Theorem 2. ([3]) Let $K \subsetneq 2^{[n]}$ be a proper simplicial complex such that $Vert(K) = [n]$. Then $W = 2^{[n]} \setminus K$ is a *roughly weighted simple game* with all weights strictly positive if and only if the canonical fan $Fan(K)$ of W is *pseudo-polytopal* in the sense that it refines the normal fan of a convex polytope.

Conclusion: Roughly weighted simple games correspond to pseudo-polytopal Bier spheres!

Generalized permutahedra

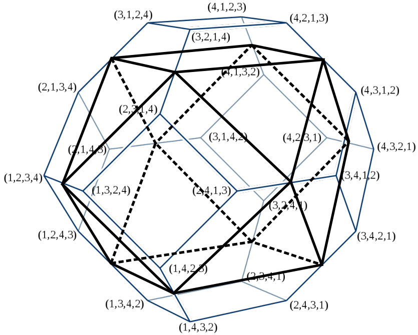


Figure: Root polytope $Root_4$ inscribed in the regular permutohedron $Perm_4$.

Steinitz problem

There are (F. Lutz, 2007) 247882 combinatorial spheres with 10 vertices and the problem of deciding which of them are (non)polytopal is still wide open!?

All simplicial 3-spheres with up to 7 vertices are polytopal, and only two 3-spheres with 8 vertices are nonpolytopal, the Grünbaum-Sreedharan sphere and the Barnette sphere.

The classification of triangulated 3-spheres with 9 vertices into polytopal and nonpolytopal spheres was started by Altshuler and Steinberg and completed by Altshuler, Bokowski, and Steinberg, etc.

Steinitz problem for Bier spheres

The following theorem is our main new experimental result.

Theorem 3. ([3]) All Bier spheres with up to eleven vertices are polytopal, in particular this holds for all 3-dimensional Bier spheres. For illustration, there are 88 non-threshold complexes on 5 vertices and 48 corresponding non-isomorphic Bier spheres. Explicit convex realizations of all spheres with 10 and 11 vertices can be respectively found in

https://imi.pmf.kg.ac.rs/pub/m_timotijevic/bier_kv5_d3.pdf
https://imi.pmf.kg.ac.rs/pub/m_timotijevic/bier_kv5_d4.pdf

Towards the proof of Theorem 3

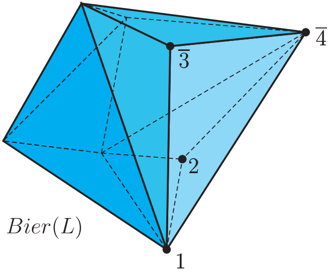
Theorems 1 and 2 provide a complete characterization of (roughly) weighted games (threshold complexes) in terms of the canonical polytopality (pseudo-polytopality) of the corresponding Bier spheres.

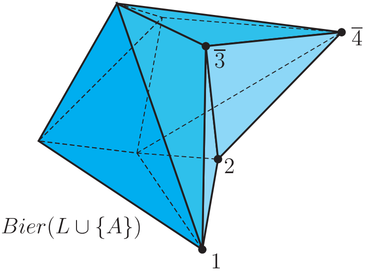
It is known that with the increase of the number of vertices (number of players) Bier spheres tend to be nonpolytopal.

Theorem 3, and the corresponding algorithm for proving polytopality of Bier spheres, show that nonpolytopal Bier spheres must have at least 12 vertices. In particular the "Möbius Bier sphere" is polytopal, although (in light of Theorem 1) it is not canonically polytopal.

Algorithm

Idea: The algorithm tries to find an explicit polytopal realization of a given Bier sphere *Bier*(K) by a sequence of modifications, where the initial step is the canonical polytopal realization of the Bier sphere *Bier*(L) of a threshold complex L (chosen to be as close to K as possible).





Radial variation of vertices

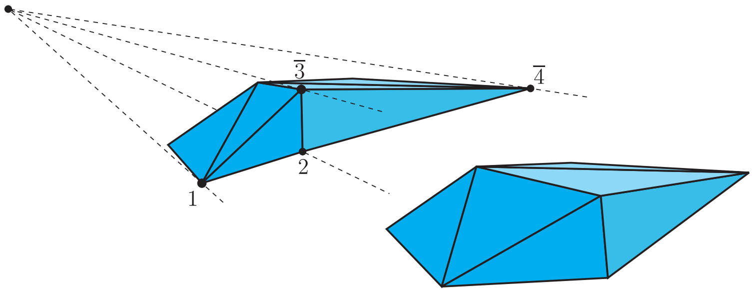
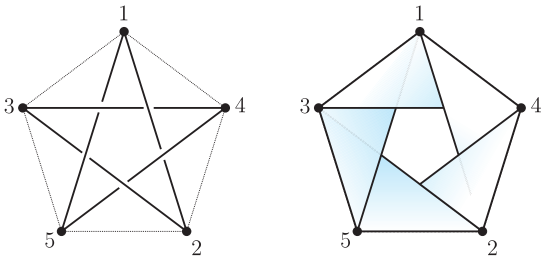


Figure: Radial variation of vertices.

Möbius sphere



Let $K_1 = L \cup \{\{1,5\}\}$, $K_2 = K_1 \cup \{\{3,4\}\}$ and $K = K_3 = K_2 \cup \{\{4,5\}\}$ be a sequence of successive approximations $L \subset K_1 \subset K_2 \subset K$, where $L = 2^{\{1,2\}} \cup 2^{\{2,3\}} \cup \{\{4\},\{5\}\}$.

Convex realization of $Bier(L)$

L is a threshold complex with weights $\mu = \{\frac{3}{10}, \frac{1}{50}, \frac{1}{25}, \frac{8}{25}, \frac{8}{25}\}$ and the threshold $\alpha = \frac{33}{100}$. $Bier(L)$ has a convex realization with vertices listed as the rows of the following matrix

$$V_{Bier(L)} = \begin{bmatrix} \frac{10}{3} & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & \frac{25}{8} \\ -\frac{25}{8} & -\frac{25}{8} & -\frac{25}{8} & -\frac{25}{8} \\ -\frac{670}{99} & 0 & 0 & 0 \\ 0 & -\frac{3350}{33} & 0 & 0 \\ 0 & 0 & -\frac{1675}{33} & 0 \\ 0 & 0 & 0 & -\frac{1675}{264} \\ \frac{1675}{264} & \frac{1675}{264} & \frac{1675}{264} & \frac{1675}{264} \end{bmatrix}$$

Convex realization of $Bier(K_1)$

$$V_{Bier(K_1)} = \begin{bmatrix} 3.37502 & -5.09311 & -2.85102 & -0.896848 \\ 0.0296025 & 50.7462 & 0.155815 & -0.368407 \\ 0.0296025 & 0.746166 & 25.1558 & -0.368407 \\ 0.0296025 & 0.746166 & 0.155815 & 2.75659 \\ -3.07544 & -12.7875 & -8.32932 & -4.43607 \\ -6.73807 & 0.746166 & 0.155815 & -0.368407 \\ -0.0659424 & -52.6647 & 24.8672 & 3.97454 \\ 0.0087548 & 11.2174 & -45.1973 & 0.579216 \\ 0.0325764 & -0.747559 & -0.613352 & -6.84851 \\ 6.3743 & 7.09086 & 6.50051 & 5.97629 \end{bmatrix};$$

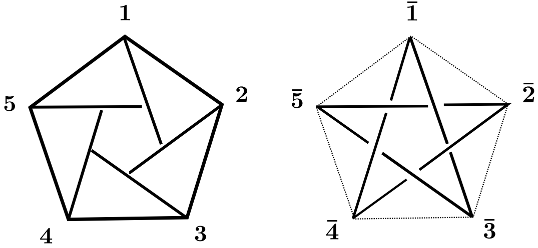
Convex realization of $Bier(K_2)$

$$V_{Bier(K_2)} = \begin{bmatrix} 3.38521 & -3.86036 & -4.47002 & -1.2378 \\ 0.0397926 & 51.9789 & -1.46319 & -0.709358 \\ -0.0109639 & -3.9592 & 31.3401 & 0.930843 \\ 0.0118228 & -1.29333 & 2.82878 & 3.3205 \\ -3.06525 & -11.5548 & -9.94833 & -4.77702 \\ -6.71696 & 3.16451 & -3.01825 & -1.03684 \\ -0.0557521 & -51.432 & 23.2482 & 3.63359 \\ 0.0189449 & 12.4502 & -46.8163 & 0.238265 \\ 0.0427665 & 0.485194 & -2.23236 & -7.18946 \\ 6.35039 & 4.02084 & 10.5314 & 6.82728 \end{bmatrix};$$

Convex realization of $Bier(K_3)$

$$V_{Bier(K_3)} = \begin{bmatrix} 2.58418 & -7.84043 & -7.2259 & -1.12488 \\ -0.761242 & 47.9988 & -4.21908 & -0.59644 \\ -0.811999 & -7.93927 & 28.5842 & 1.04376 \\ -0.538235 & -4.02784 & 0.93516 & 3.39776 \\ -2.77768 & -10.1347 & -8.96312 & -4.81456 \\ -7.23817 & 0.568376 & -4.81495 & -0.963001 \\ 2.40876 & -39.1324 & 31.681 & 3.28216 \\ 2.3433 & 23.9615 & -38.7646 & -0.0884603 \\ -0.758268 & -3.49488 & -4.98824 & -7.07654 \\ 5.54935 & 0.0407717 & 7.77551 & 6.9402 \end{bmatrix}$$

Is this really a convex realization of the Möbius sphere?



$$
\begin{array}{cccccc}
1\bar{2}3\bar{4} & \bar{2}3\bar{4}5 & 3\bar{4}5\bar{1} & \bar{4}5\bar{1}2 & 5\bar{1}2\bar{3} & + \\
\bar{1}2\bar{3}4 & 2\bar{3}4\bar{5} & \bar{3}4\bar{5}1 & 4\bar{5}1\bar{2} & \bar{5}1\bar{2}3 & + \\
\bar{1}23\bar{4} & \bar{2}34\bar{5} & \bar{3}45\bar{1} & \bar{4}51\bar{2} & \bar{5}12\bar{3} & + \\
123\bar{4} & 234\bar{5} & 345\bar{1} & 451\bar{2} & 512\bar{3} & - \\
\bar{1}234 & \bar{2}345 & \bar{3}451 & \bar{4}512 & \bar{5}123 & - \, .
\end{array} \tag{5}
$$

Comparison of lists

The table (5) lists all facets of the Möbius sphere.
Polymake (E. Gawrilow and M. Joswig), applied to the matrix $V_{Bier(K_3)}$ produces the following list of facets of $Q = Conv(V_{Bier(K_3)})$:

$$Facets(Q) = \left\{ \begin{array}{ccccc} 1258 & 1578 & 0178 & 4578 & 2356 \\ 3467 & 2568 & 4568 & 3456 & 0468 \\ 0478 & 0467 & 3457 & 3579 & 0679 \\ 0689 & 2689 & 3679 & 2359 & 2369 \\ 1289 & 1259 & 1579 & 0189 & 0179 \end{array} \right\} \qquad (6)$$

The lists (5) and (6) are isomorphic!

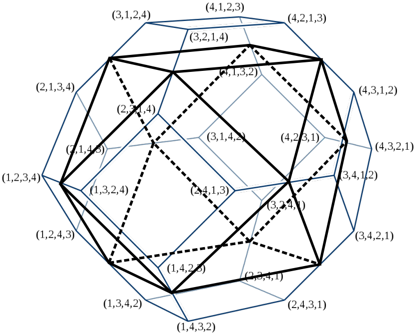


Figure: Root polytope $Root_4$ inscribed in the regular permutohedron $Perm_4$.