Quantum Dynamics as Classical Dynamics and Quantum/Symplectic Computers

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Outlook

- **Quantum-symplectic duality**
  - Schrodinger equation and symplectic mechanics
  - Unitary and symplectic evolution
  - Qubits and Symbits
  - Euler-Bernoulli Eq. Elasticity theory. Vibrating beam, plates.
- **Quantum/Symplectic computer**
  - Unitary group $U(N)$ and symplectic group $Sp(2N, \mathbb{R})$
  - NOT and $\gamma(\text{NOT})$ gates
- **Conclusion**

Quantum-symplectic duality

Schrodinger equation and symplectic mechanics

- Main message: Schrodinger equation actually describes the classical symplectic Hamiltonian system.
- Euler-Bernoulli Eq. Elasticity theory. Vibrating beam, plates
- Quantum/Symplectic computer
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- Compare:

  wave-particle duality Einstein(photons)-de Broglie(matter waves),...

Schrodinger equation and symplectic mechanics

- **Hilbert space**

$$\mathcal{H} = \mathbb{C}^{\mathbb{N}} \tag{1}$$

- **Schrodinger equation (SE)**

$$\mathbf{i}\dot{\psi}_{\mathbf{a}} = \mathbf{H}_{\mathbf{ab}}\psi_{\mathbf{b}}$$

$$\mathbf{a}, \mathbf{b} = \mathbf{1}...\mathbf{N}, \psi_{\mathbf{a}} \in \mathbb{C}, \quad \mathbf{H}_{\mathbf{ab}} = \overline{\mathbf{H}}_{\mathbf{ba}} \tag{2}$$

Schrodinger equation in real variables

- $\psi_{\mathbf{a}} = \mathbf{q_a} + \mathbf{ip_a}$, where q_a, p_a are real.
- $H_{ab} = K_{ab} + iL_{ab}$,
- $K_{ab} = K_{ba}$, $L_{ab} = -L_{ba}$ **matrices with real entries.**
- **Then SE:**

$$\dot{\mathbf{q}}_{\mathbf{a}} = \mathbf{K_{ab}p_b} + \mathbf{L_{ab}q_b} \tag{3}$$

$$\dot{\mathbf{p}}_{\mathbf{a}} = -\mathbf{K_{ab}q_b} + \mathbf{L_{ab}p_b}(*) \tag{4}$$

Schrodinger equation in real variables

- **(*) are classical symplectic Hamiltonian equations**

$$\dot{q}_a = \frac{\partial H_{sym}}{\partial p_a},$$
$$\dot{p}_a = -\frac{\partial H_{sym}}{\partial q_a}, \qquad (5)$$

with the Hamiltonian

$$H_{sym} = \frac{1}{2}(p_a K_{ab}\, p_b + q_a K_{ab}\, q_b) + p_a L_{ab}\, q_b$$

Remark 1.

If the Hamiltonian equation in the Schrodinger equations depends on time

$$\mathbf{i}\dot{\psi}_{\mathbf{a}} = \mathbf{H_{ab}(t)}\psi_{\mathbf{b}} \tag{6}$$

with

$$\mathbf{H_{ab}(t) = K_{ab}(t) + iL_{ab}(t)}, \tag{7}$$

then we get the following equation

$$\mathbf{\dot{q}_a = K_{ab}(t)p_b + L_{ab}(t)q_b}$$
$$\mathbf{\dot{p}_a = -K_{ab}(t)q_b + L_{ab}(t)p_b}$$

the Hamiltonian equation

$$\mathbf{H_{sym}(t) = \frac{1}{2}(p_a K_{ab}(t)\, p_b + q_a K_{ab}(t)\, q_b) + p_a L_{ab}\, q_b}.$$

Schrodinger equation

$$\mathbf{i}\dot{\psi} = \mathbf{H}\psi \tag{8}$$

where $\mathbf{H} = \frac{\mathbf{d^2}}{\mathbf{dx^2}} + \mathbf{V(x)}$

$$\ddot{\mathbf{\Psi}} + \mathbf{H^2\Psi} = \mathbf{0} \tag{9}$$

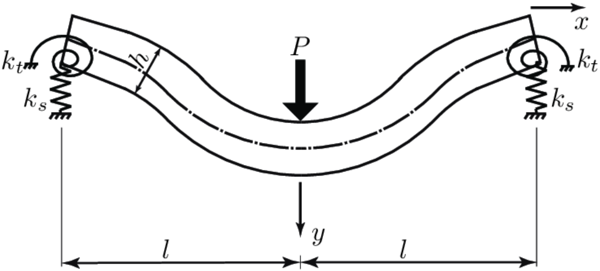
$$\mathbf{\Psi} = \mathbf{q} + \mathbf{ip} \tag{10}$$

$$\ddot{\mathbf{q}} + \mathbf{H^2 q} = \mathbf{0}, \qquad \mathbf{q} = \mathbf{q(x,t)} \tag{11}$$

Elasticity theory

- Euler-Bernoulli vibrating beam eq.

$$\rho\ddot{\mathbf{W}} + \frac{\mathbf{d^2}}{\mathbf{dx^2}}(\mathbf{K}(\mathbf{x})\frac{\mathbf{d^2}}{\mathbf{dx^2}}\mathbf{W}) = \mathbf{0} \tag{12}$$



Vibrating plates

$$\rho\ddot{\mathbf{y}} + \mathbf{K_1}\boldsymbol{\Delta}\mathbf{y} + \mathbf{K_2}\boldsymbol{\Delta}\boldsymbol{\Delta}\mathbf{y} = \mathbf{0} \tag{13}$$

- **Euler-Bernoulli 1650 year beam**
- **Eiffel Tower**

$$\ddot{\mathbf{W}} + \frac{\mathbf{d^4}}{\mathbf{dx^4}}\mathbf{W} = \mathbf{0} \tag{14}$$

- **Schrodinger equation for free particle**

$$\mathbf{i}\dot{\boldsymbol{\Psi}} = -\frac{\mathbf{d^2}}{\mathbf{dx^2}}\boldsymbol{\Psi} \tag{15}$$

Symplectic space

- **Symplectic space is a pair** (V, ω)
- V **- real vector space,** ω **q-summetric non-degenerate bilinear form**
- **Symplectic group is group if linear transformation** (V) **which preserves** ω
- **Example:** $\mathbb{R}(2N)$ **there exist the standart symplectic structure which is given by the matrix**

$$\begin{pmatrix} \mathbf{0} & \mathbf{iN} \\ -\mathbf{iN} & \mathbf{0} \end{pmatrix} \tag{16}$$

- **The correcsponding symplectic group is denote** $\mathbf{Sp}(\mathbf{2N}, \mathbb{R})$

Unitary and symplectic evolution

- **Relation between representations of solutions of the Schrodinger/Hamiltonian equations in the complex unitary form and the real symplectic form.**
- **The solution of the N-component Schrodinger equation**

$$\psi(\mathbf{t}) = \mathbf{U_t}\,\psi(\mathbf{0}), \qquad \psi(\mathbf{t}) = (\psi_{\mathbf{1}}(\mathbf{t}), ...\psi_{\mathbf{N}}(\mathbf{t}))\,,$$

in term of real N-component vectors $\psi(\mathbf{t}) = \mathbf{q}(\mathbf{t}) + \mathbf{i}\mathbf{p}(\mathbf{t})$.

Unitary and symplectic evolution

- **Representing the unitary matrix U_t in the form $\mathbf{U_t = X_t + iY_t}$, where X_t and Y_t are real matrices, we have**

$$\psi(\mathbf{t}) = \mathbf{X_t q(0) - Y_t p(0) + i(Y_t q(0) + X_t p(0))},$$

i.e. $\quad \mathbf{q(t) = X_t q(0) - Y_t p(0)}, \qquad \mathbf{p(t) = Y_t q(0) + X_t p(0)} \quad (\#)$

- **Relations** $(\#)$ **can be represented in the form**

$$\begin{pmatrix} \mathbf{q(t)} \\ \mathbf{p(t)} \end{pmatrix} = \mathbf{S_t} \begin{pmatrix} \mathbf{q(0)} \\ \mathbf{p(0)} \end{pmatrix}, \quad \mathbf{S_t} = \begin{pmatrix} \mathbf{X_t} & \mathbf{Y_t} \\ \mathbf{-Y_t} & \mathbf{X_t} \end{pmatrix}$$

Unitary and symplectic evolution

- **The unitarity of matrix U_t,**

$$\mathbf{U_t^* U_t} = (\mathbf{X_t^T} - \mathbf{i}\mathbf{Y_t^T})(\mathbf{X_t} + \mathbf{i}\mathbf{Y_t}) \tag{17}$$

$$= \mathbf{X_t^T X_t} + \mathbf{Y_t^T Y_t} - \mathbf{i}(\mathbf{Y_t^T X_t} - \mathbf{X_t^T Y_t}) = \mathbf{I} \tag{18}$$

means that

$$\mathbf{X_t^T X_t} + \mathbf{Y_t^T Y_t} = \mathbf{I},$$

$$\mathbf{Y_t^T X_t} - \mathbf{X_t^T Y_t} = \mathbf{0}\ (*)$$

Unitary and symplectic evolution

- **Indeed,**

$$\mathbf{S_t^T J S_t} = \begin{pmatrix} \mathbf{X_t^T Y_t - Y_t^T X_t} & \mathbf{Y_t^T Y_t + X_t^T X_t} \\ \mathbf{-Y_t^T Y_t - X_t^T X_t} & \mathbf{-Y_t^T X_t + X_t^T Y_t} \end{pmatrix}$$

and taking into account (*) we get

$$\mathbf{S_t^T J S_t} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{-I} & \mathbf{0} \end{pmatrix}, \quad \textbf{i.e.}\,(**)$$

Qubits and Symbits

- **In quantum computing, the fundamental unit of information is the qubit.**
- **The qubit is a two-level quantum system, mathematically represented by the complex Hilbert space $\mathbb{C}^2$.**
- **A general state of a qubit can be written as:**

$$|\psi\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^{\mathbf{2}} + |\beta|^{\mathbf{2}} = \mathbf{1},$$

$|\mathbf{0}\rangle$ **and** $|\mathbf{1}\rangle$ **form an orthonormal basis for** $\mathbb{C}^{\mathbf{2}}$

$$\alpha, \beta \in \mathbb{C}$$

Qubits and Symbits

- **In the symplectic computing framework, the analogous concept to a qubit is a symplectic bit (symbit).**
- **A symbit is represented by a two-dimensional real vector space $\mathbb{R}^2$.**
- **A general state of a symbit can be written as:**

$$|\phi\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle, \quad \alpha, \beta \in \mathbb{R}, \quad \alpha^{\mathbf{2}} + \beta^{\mathbf{2}} = \mathbf{1}, \tag{19}$$

  where $|0\rangle$ and $|1\rangle$ form an orthonormal basis for $\mathbb{R}^2$.
- **The coefficients α and β here are real numbers such that the total probability is equal to one.**
- **We can interpret $\mathbb{R}^2$ as the phase plane for a dynamical system with coordinates p and q.**

Quantum mechanics over real number and Kahler space

- If $\langle .,. \rangle$ is inner product in the complex Hilbert space then

$$[\langle .,. \rangle] = (\langle .,. \rangle) + \mathbf{i}\omega \tag{20}$$

- where $(\langle .,. \rangle)$ is positive defined and ω skew-symmetric
- Kahler space (V, J, ω). V is a real vector space, J is a complex structure, J^2=-1 and $\omega(., J.)$ is positive defined.

Tensor products of N sbits

- **Tensor products of N sbits:**

$$\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \cdots \otimes \mathbb{R}^2 = \mathbb{R}^{2^N}. \tag{21}$$

  This is equivalent to $\mathbb{C}^N$.

- **On the other hand, the tensor product of N qubits is $\mathbb{C}^{2^N}$.**
- **Therefore, by using tensor products of sbits, one can obtain more general spaces than those obtained from tensor products of qubits.**

Tensor products of N sbits

- **Operations on symbits are performed using symplectic transformations, which are elements of the symplectic group $Sp(2,\mathbb{R})$.**
- **For instance, the simplest symplectic transformation in $\mathbb{R}^2$ can be represented by a 2×2 matrix S that preserves the symplectic form:**

$$S^T J S = J, \quad \textbf{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An example of a symplectic transformation is the rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

which preserves the symplectic form and hence is a member of $Sp(2,\mathbb{R})$.

Tensor products of N sbits

- **In summary, while qubits are the basic units of quantum information in $\mathbb{C}^2$ and are manipulated using unitary transformations, symbits are the basic units of symplectic information in $\mathbb{R}^2$ and are manipulated using symplectic transformations. This duality provides a bridge between quantum and symplectic computing, offering a new perspective on computational processes.**

Quantum/Symplectic computer

- **Quantum computer is a sequence of unitary transformations (gates) and projectors (measurements)**
- **Symplectic computer is a sequence of symplectic transformations and projectors**
- $$U(N) = Sp(2N, \mathbb{R}) \bigcap O(2N, \mathbb{R}) \tag{22}$$

- **Unitary group $U(N)$ and symplectic group $Sp(2N,\mathbb{R})$**

Let us show that unitary group $U(N)$ is a subgroup of symplectic group $Sp(2N,\mathbb{R})$. Let us remind that a symplectic matrix S of symplectic group $Sp(2N,\mathcal{R})$ satisfies the relation

$$S^T JS = J \tag{23}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{24}$$

Unitary and symplectic gates

- **Unitary group $U(N)$ and symplectic group $Sp(2N, \mathbb{R})$**

Let S is a $2N \times 2N$ matrix in the form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $N \times N$ matrices. The conditions on A, B, C and D are

- $A^{\mathrm{T}}C, B^{\mathrm{T}}D$ **symmetric, and** $A^{\mathrm{T}}D - C^{\mathrm{T}}B = I$
- AB^{T}**,** CD^{T} **symmetric, and** $AD^{\mathrm{T}} - BC^{\mathrm{T}} = I$

- **Let us show that there is a canonical mapping γ of the unitary group $U(N)$ to $Sp(2N,\mathbb{R})$. Let $V \in U(N)$, we present V in the form**

$$V = X + iY, \quad \text{where } X, Y \text{ are } N \times N \text{ with real entrie} \tag{25}$$

We define γ by the following formula

$$\gamma(V) = \gamma(X + iY) = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \tag{26}$$

One can check that the unitarity conditions $VV^* = V^*V = 1$ lead to the conditions for the symplectic matrices.

Remarks

- *Remark 2* **One can also check that there is a relation $\gamma(V_1)\gamma(V_2) = \gamma(V_1 V_2)$.**
- *Remark 3.* **The Schrodinger equation in the form**

$$i\dot{\psi} = H\,\psi, \tag{27}$$

where

$$H = -\Delta + V(x) \tag{28}$$

Δ is the Laplace operator in $\mathbb{R}$, in the real formulation looks as follows

$$\dot{q} = Hp, \qquad \dot{p} = -Hq. \tag{29}$$

Spectrum and scattering theory for in this approach are considered in [?]

NOT and γ(NOT) gates

- **Quantum computation is a sequence of unitary matrices (gates) of the simple form. We put into correspondence into such unitary gate a corresponding symplectic gate using the γ defined by (26).**
- **See the simplest gate is NOT. It is a unitary matrix $NOT = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ defined on the qubit $\mathbb{C}^2$. By using the mapping γ we define the corresponding symplectic gate**

$$\gamma(NOT) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{30}$$

One can construct the symplectic analog of the controlling CNOT gate.

Conclusions

- **Quantum-Symplectic duality**
- **Schrodinger equation in is equivalent to Euler-Bernauli equation for vibrating beam and elasticity theory.**
- **Symplectic computer is proposed**