UV-IR connections in scattering amplitudes:

a power of unitarity and causality

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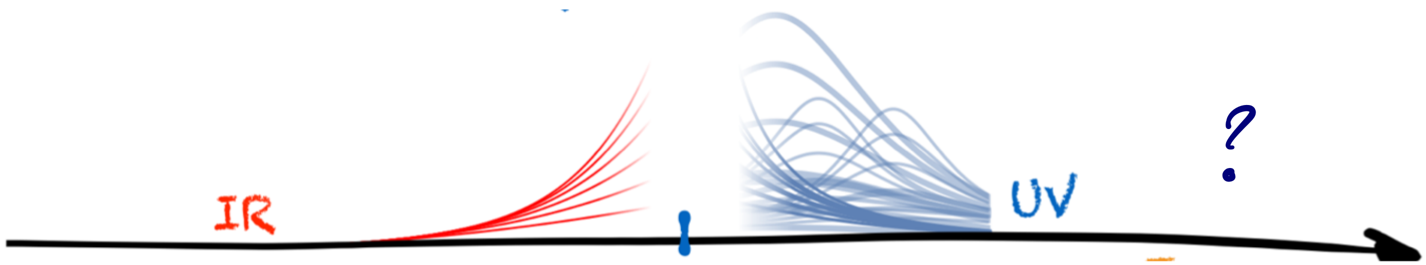
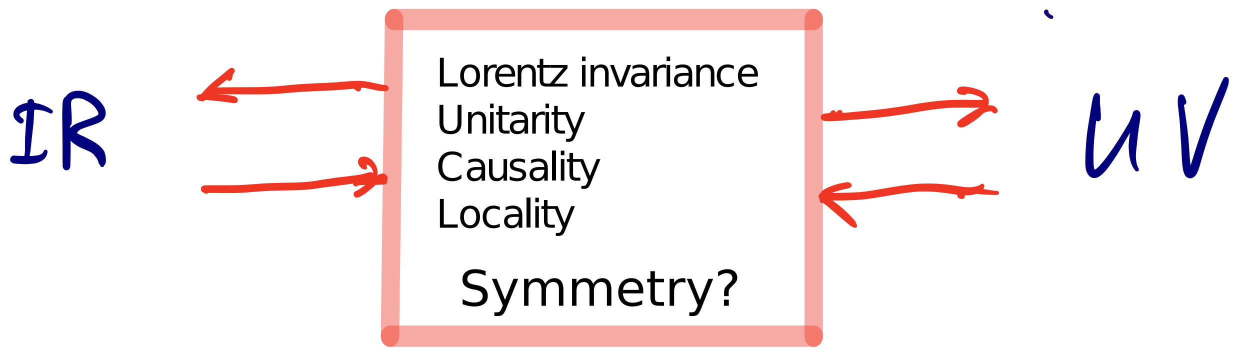
EFT framework: UV - IR connections





- Assumptions about UV constraints on IR (positivity bounds)
- IR results may require special UV properties for consistency
- The symmetry working in UV and IR can constrain the structure of IR EFT

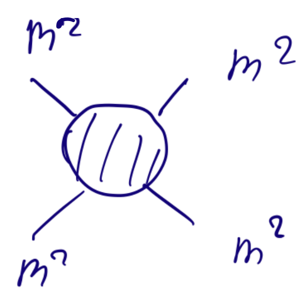
A 'good' UV completion

What do we mean by 'good'?

- Lorenz-invariant $\Rightarrow \mathcal{A} = \mathcal{A}(s, t, u)$
- unitary $\Rightarrow Im\, \mathcal{A} > 0$
- satisfying causality $\Rightarrow \mathcal{A}(s, t, u)$ is analytic everywhere except real axes
- local $\Rightarrow$ polynomial boundedness (Froissart-Martin bound)

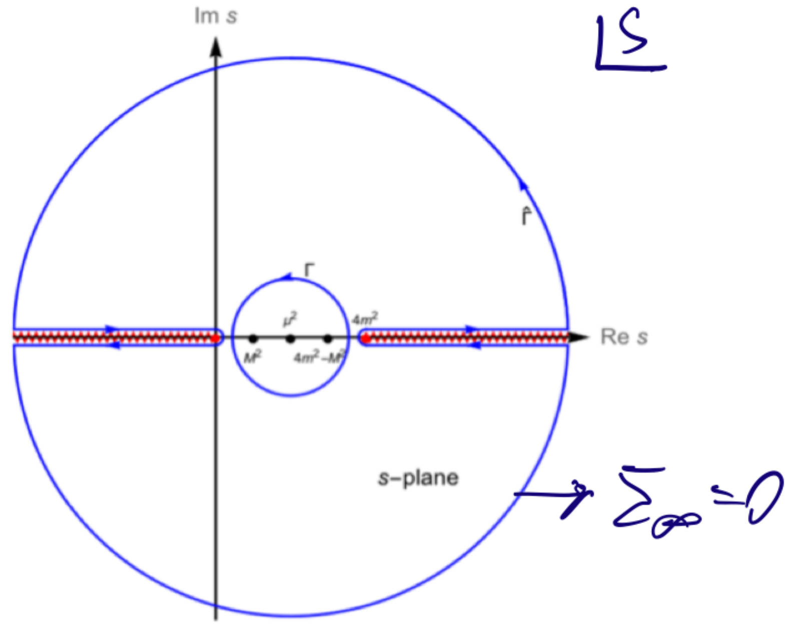
$$A(s) < s \log^2 s$$

$2 \to 2$ amplitude



What is positive in positivity bounds?

Example: forward limit $t = 0$



Singularities:

- poles on real axes

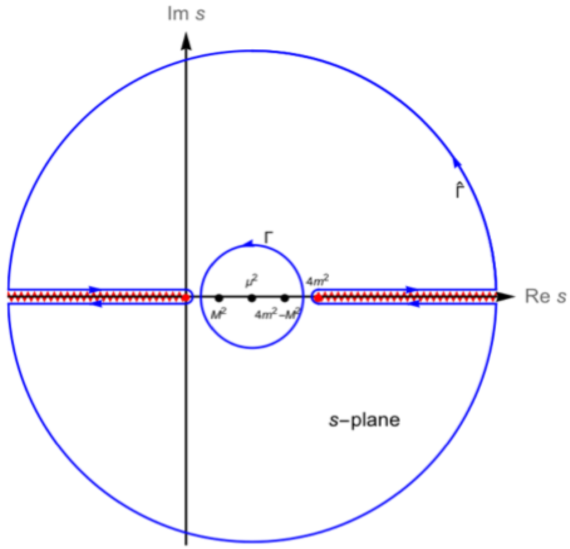
- branch cuts

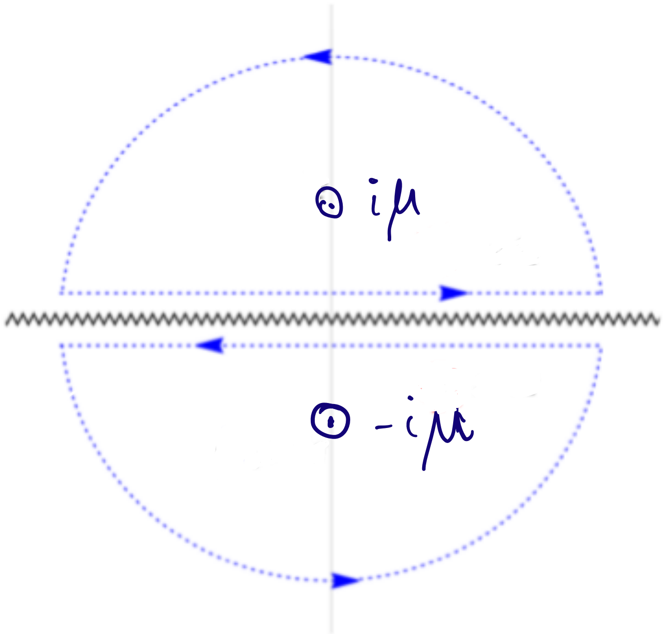
$$\Sigma_{IR} = \frac{1}{2\pi i}\int_\Gamma ds \frac{\mathcal{A}(s)}{(s-\mu^2)^3} = \int_{4m^2}^{\infty} \frac{ds}{\pi}\left(\frac{Im\mathcal{A}(s)}{(s-\mu^2)^3} + \frac{Im\mathcal{A}^{+}(s)}{(s-4m^2+\mu^2)^3}\right)$$

$$\Sigma_{IR} = \frac{1}{2}\mathcal{A}''(\mu^2) > 0$$

Positivity bounds: massive vs massless

Dispersive relations



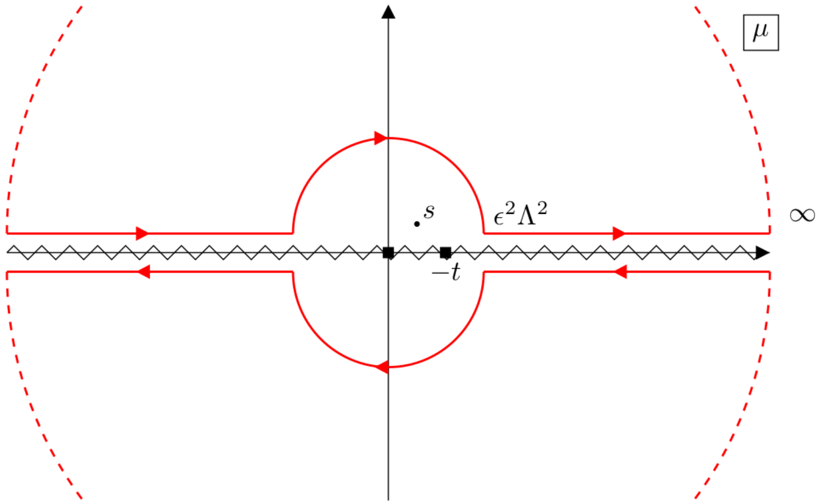


$$\Sigma_m = \frac{1}{2}A_{ss}(\mu^2) = \frac{1}{2\pi i}\int_\Gamma \frac{A(s)ds}{(s-\mu^2)^3} = \frac{1}{\pi}\int_{4m^2}^{\infty}\left(\frac{ImA(s)ds}{(s-\mu^2)^3} + \frac{ImA_*(s)ds}{(s+\mu^2-4m^2)^3}\right)$$

$$\Sigma_0 = \frac{\mathcal{A}_{ss}(\mu^2)}{16} - \frac{3i\mathcal{A}_s(\mu^2)}{16\mu^2} = \frac{1}{2\pi i}\int_\Gamma \frac{s^3 A(s)ds}{(s^2+\mu^4)^3} = \int_0^\infty \frac{ImA(s)s^3 ds}{(s^2+\mu^4)^3} + \text{crossed}$$

Herrero-Valea, Santos-Garcia, AT'20

Advance further: non-linear bounds



$$C(n,m) \equiv \frac{n!/2}{2\pi \mathrm{i}} \int_{\text{arcs}} \mathrm{d}\mu \, \frac{\partial_t^m \mathcal{A}(\mu,0)}{\mu^{n+1}} = \frac{\partial^m}{\partial t^n}\frac{\partial^n}{\partial s^n}\mathcal{A}(s,t)$$

$$I_{s,u}(n,m) \equiv \int_{\epsilon^2\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi} \frac{\partial_t^m \mathrm{Disc}_s \mathcal{A}_{s,u}(\mu,0)}{\mu^{n+1}} > 0$$

Cauchy–Schwarz inequality

$$\left(\sum_{i=1}^n u_i v_i\right)^2 \leq \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right)$$

$(I^4)^N < (I^2)^{N-1} \cdot I^{2N+2}$ $\quad (I^4)^2 < I^6 I^2$

$(I^4)^3 < (I^2)^2 I^8$

$$I_{s,u}(3,0)^2 < I_{s,u}(2,0)I_{s,u}(4,0)$$

$C_{40}^2 < \frac{2}{5} C_{20} C_{60}$

$$\frac{4}{3}C(3,0)^2 < C(2,0)C(4,0) \Rightarrow \frac{4}{3}\mathcal{A}_{sss}^2 < \mathcal{A}_{ss}\mathcal{A}_{ssss}$$

many inequalities can be derived!

$$C(2,1) - \frac{1}{2}C(3,0) + \frac{\sqrt{3}}{4}\sqrt{C(2,0)C(4,0)} > 0\,.$$

M. Carrillo González, C. de Rham, S. Jaitly, V. Pozsgay, and A. Tokareva, (2023), arXiv:2307.04784 [hep-th].

Photon EFT and amplitudes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda^4}F^{\mu\nu}F_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + \frac{c_2}{\Lambda^4}F^{\mu\nu}F^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{c_3}{\Lambda^6}F^{\alpha\mu}F^{\nu\beta}\partial_\mu F_{\beta\gamma}\partial_\nu F_\alpha{}^\gamma + \frac{c_4}{\Lambda^6}F^{\alpha\mu}F^{\nu\beta}\partial_\beta F_{\mu\gamma}\partial^\gamma F_{\alpha\nu} + \frac{c_5}{\Lambda^6}F^{\alpha\mu}F^{\nu\beta}\partial_\beta F_{\nu\gamma}\partial^\gamma F_{\alpha\mu} + \frac{c_6}{\Lambda^8}F^{\mu\nu}\partial_\mu F_{\nu\rho}\partial^\rho\partial^\alpha F^{\beta\gamma}\partial_\alpha F_{\beta\gamma} + \frac{c_7}{\Lambda^8}F^\mu{}_\gamma\partial_\mu F_{\nu\rho}\partial^\nu F_{\alpha\beta}\partial^\rho\partial^\gamma F^{\alpha\beta} + \frac{c_8}{\Lambda^8}F^{\mu\gamma}\partial_\mu F_{\nu\rho}\partial^\rho\partial^\beta F_{\alpha\gamma}\partial^\alpha F^\nu{}_\beta\,.$$

$$\mathcal{A}_{++++} = f_2\left(s^2+t^2+u^2\right) + f_3 stu + f_4\left(s^2+t^2+u^2\right)^2$$
$$\mathcal{A}_{++--} = g_2 s^2 + g_3 s^3 + g_4 s^4 + g_4' s^2 tu$$
$$\mathcal{A}_{+++-} = h_3 stu$$

$$f_2 = 2\left(4c_1+c_2\right)\,,\quad g_2 = 2\left(4c_1+3c_2\right)$$
$$f_3 = -3\left(c_3+c_4+c_5\right)\,,\quad g_3 = -c_5\,,\quad h_3 = -\frac{3}{2}c_3\,,$$
$$f_4 = \frac{1}{4}c_6\,,\quad g_4 = \frac{1}{2}(c_6-c_8)+c_7\,,\quad g_4' = -\frac{1}{2}(c_7+c_8)\,.$$

$$\mathcal{A}_u(s,t,u) = \sum_{h_i}\alpha_{h_1}\beta_{h_2}\alpha^*_{-h_3}\beta^*_{-h_4}\mathcal{A}_{h_1h_4h_3h_2}(s,t,u) = \sum_{h_i}\alpha_{h_1}\beta^*_{-h_2}\alpha^*_{-h_3}\beta_{h_4}\mathcal{A}_{h_1h_2h_3h_4}(s,t,u)$$

$$\alpha_+ = \cos\theta\,,\quad \alpha_- = \sin\theta\, e^{i\phi}\,,\quad \beta_+ = \cos\chi\,,\quad \beta_- = \sin\chi\, e^{i\psi}\,.$$

Indefinite polarisation scattering

$$\mathcal{A}_{\text{ih}} = \frac{1}{2}(\cos(2\theta)(\mathcal{A}_{++--} - \mathcal{A}_{+--+})\cos(2\chi) + \mathcal{A}_{++--} + 4\mathcal{A}_{+---}\sin(\chi)\cos(\chi)\cos(\psi) + \mathcal{A}_{+--+} + \sin(2\theta)\sin(2\chi)(\mathcal{A}_{++++}\cos(\psi+\phi) + \mathcal{A}_{+-+-}\cos(\phi-\psi)) + 4\mathcal{A}_{+---}\sin(\theta)\cos(\theta)\cos(\phi))\,.$$

Linear bounds

$$g_2 + f_2 \sin(2\theta)\sin(2\chi)\cos(\psi+\phi) > |g_3\cos(2\theta)\cos(2\chi)| \,,$$

$$6g_2 > 6g_3 + 8h_3\sin(\chi)\cos(\chi)\cos(\psi) + 8h_3\sin(\theta)\cos(\theta)\cos(\phi) + \sin(2\theta)\sin(2\chi)\cos(\psi+\phi)(-6f_2+2f_3)$$

$$f_2\sin(2\theta)\sin(2\chi)\cos(\psi+\phi) + g_2 > 2f_4\sin(2\theta)\sin(2\chi)\cos(\psi+\phi) + g_4 > 0\,,$$

$$\sin(2\chi)(2\sin(2\theta)(3f_2 - f_3 + 8f_4)\cos(\psi+\phi) - 4h_3\cos(\psi)) + 6g_2 + 8g_4 > 6g_3 + 4(2g_4 + g_4')\cos(2\theta)\cos(2\chi) + 4h_3\sin(2\theta)\cos(\phi)\,.$$

Should be valid for any values of angles

Non-linear bounds

$$(g_2 + f_2\cos(\phi+\psi)\sin(2\theta)\sin(2\chi))(g_4 + 2f_4\cos(\phi+\psi)\sin(2\theta)\sin(2\chi)) > g_3^2\cos^2(2\theta)\cos^2(2\chi)$$

$$3\sqrt{(g_2 + f_2\cos(\phi+\psi)\sin(2\theta)\sin(2\chi))(g_4 + 2f_4\cos(\phi+\psi)\sin(2\theta)\sin(2\chi))} > 3g_3 + 2h_3(\sin(2\theta)\cos\phi + \sin(2\chi)\cos\psi) + f_3\cos(\phi+\psi)\sin(2\theta)\sin(2\chi)\,.$$

Analytic optimisation:

$$A_1 + A_2\cos(2\theta)\cos(2\chi) + A_3(\cos\phi\sin(2\theta) + \cos\phi\sin(2\chi)) + A_4\cos(\phi-\psi)\sin(2\theta)\sin(2\chi) + A_5\cos(\phi+\psi)\sin(2\theta)\sin(2\chi) > 0$$

$$\Leftarrow \quad A_1^2 > A_2^2 + A_3^2$$

$$\Rightarrow \quad (A_3^2 - A_4A_5)y^2 + 2(A_1A_2 - 2A_3(A_4+A_5))y + A_1^2 - A_3^2 - A_4^2 + 2A_4A_5 - A_5^2 > 0 \qquad y = \cos\psi$$

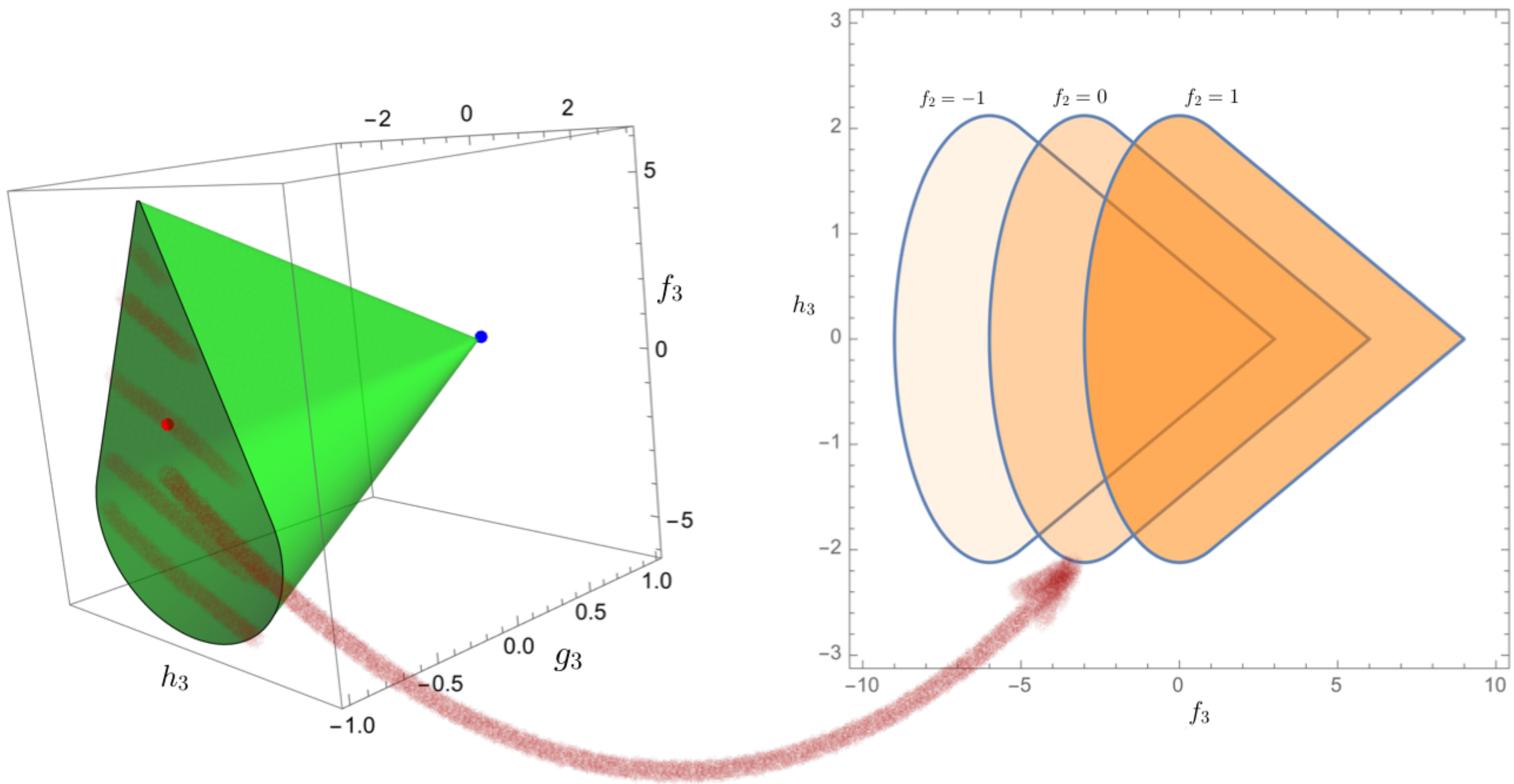
Our assumptions

Property	Causality Bounds	Positivity Bounds
Lorentz invariance	• Lorentz invariant EFT	• Invariant EFT and UV completion • Crossing symmetry
Unitarity	• Hermitian Hamiltonian: real Wilson coefficients	• Positive discontinuity of the EFT and UV amplitude
Causality	• No resolvable time advance	• Analyticity of amplitude in the complex s plane for fixed t
Locality	• IR theory is local	• IR and UV theories are local • Froissart-like bound in the UV
Other assumptions	• EFT and WKB expansions under control • Background generated by localized external source	• IR EFT is under perturbative control

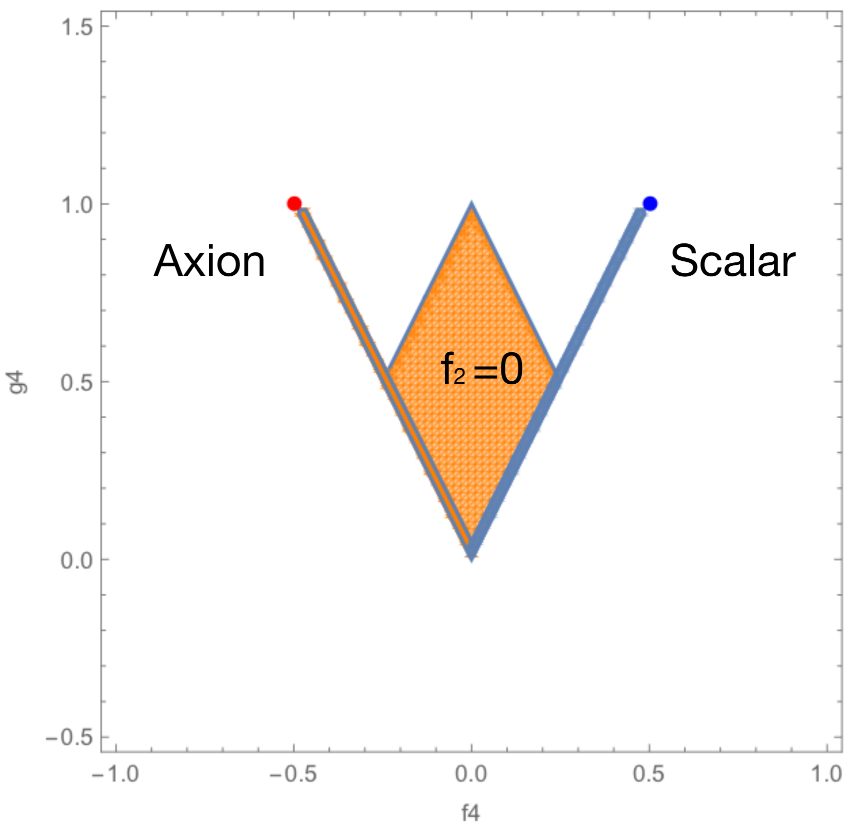
Positivity bounds

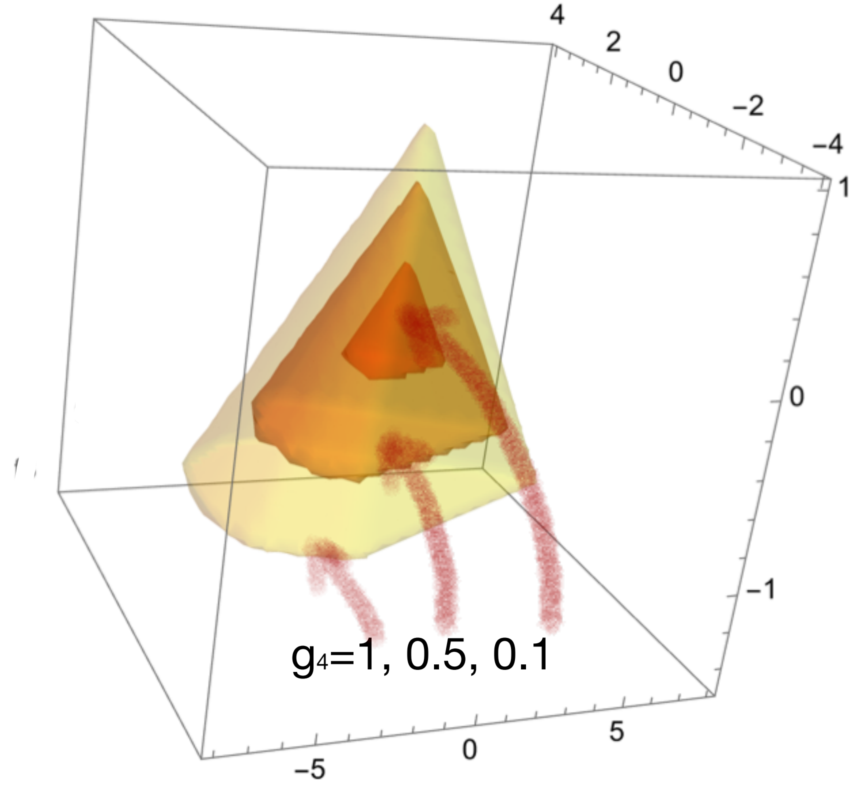
$f_2 = 0$

Many inequalities bounding 5 parameters...
We plot slices of 5D figure



Positivity bounds



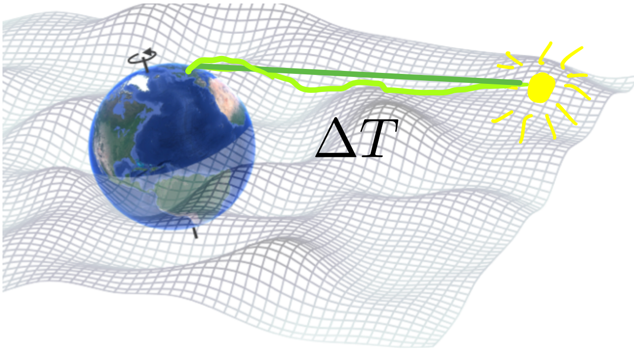


Dim 6 operators are squeezed between dim 4 and dim 8

Definitions of causality

No time machine - what does it mean?

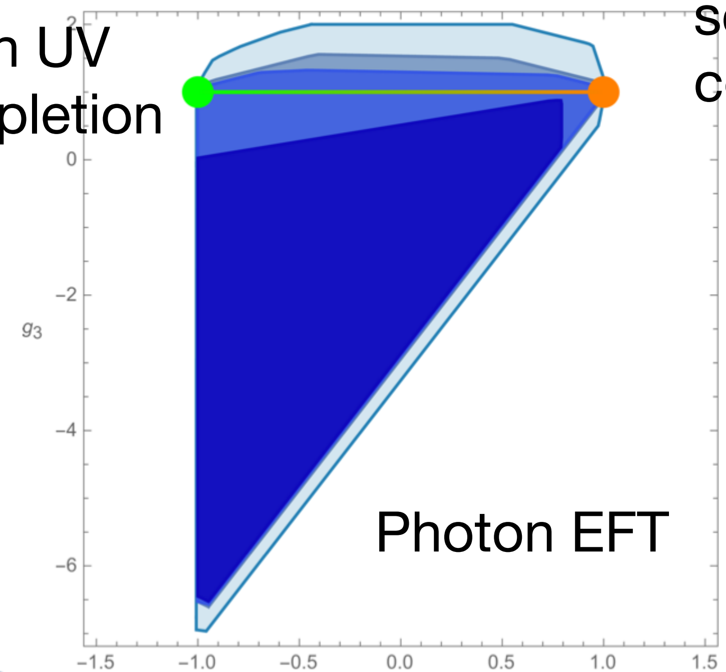
ΔT - time delay



$\Delta T > 0$ - strict causality condition

rules out all higher derivative terms

X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, *JHEP* **02** (2016) 020 [1407.5597].

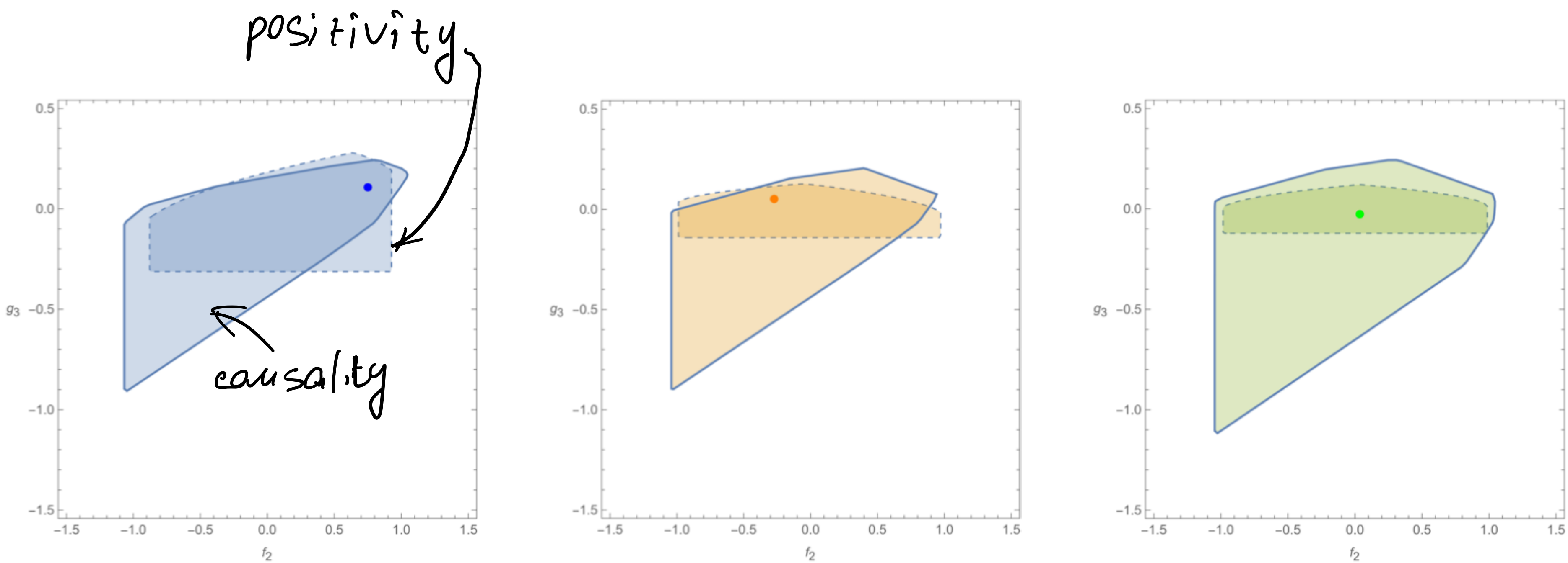


Weaker condition

$$\Delta T > -\frac{O(1)}{\omega}$$

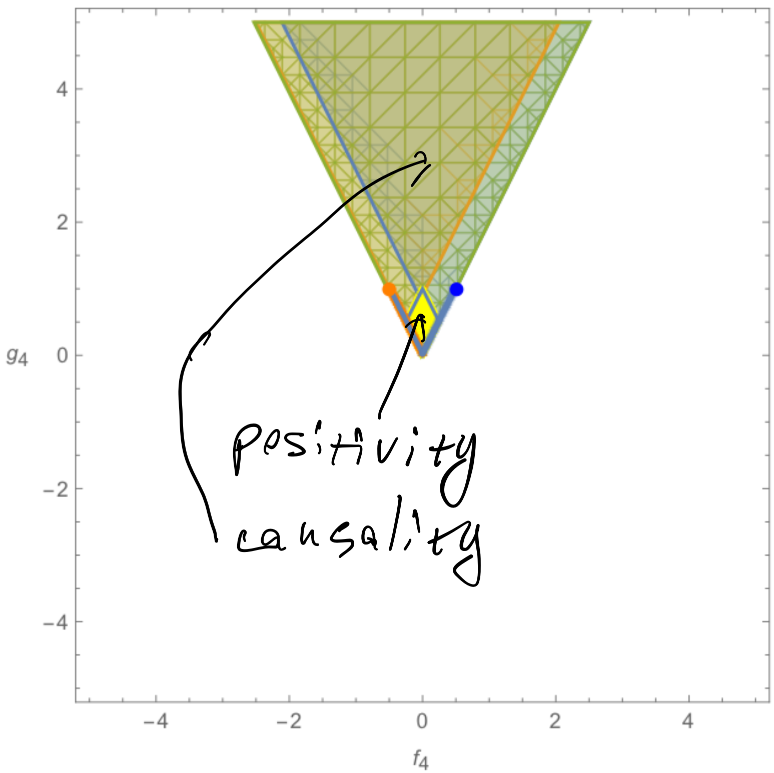
Unresolvable time advance

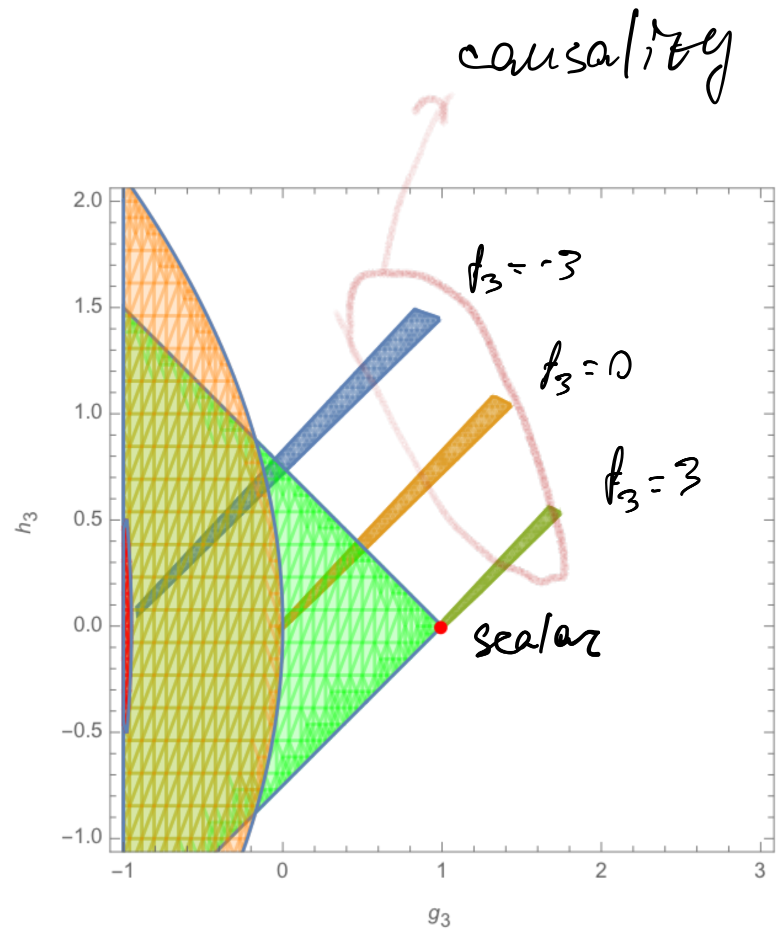
Causality vs positivity plots



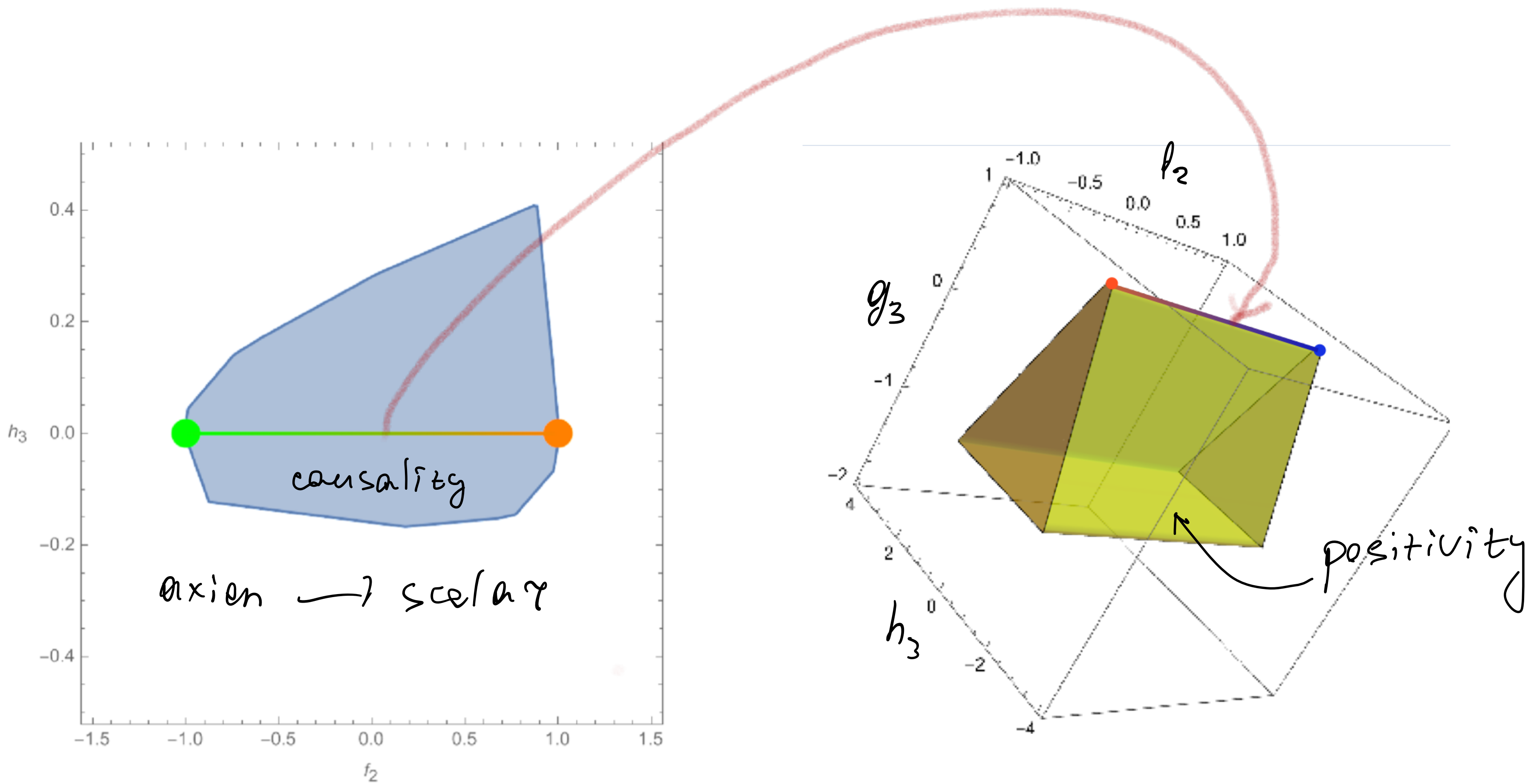
(a) Scalar (b) Spinor (c) Vector

Causality vs positivity plots



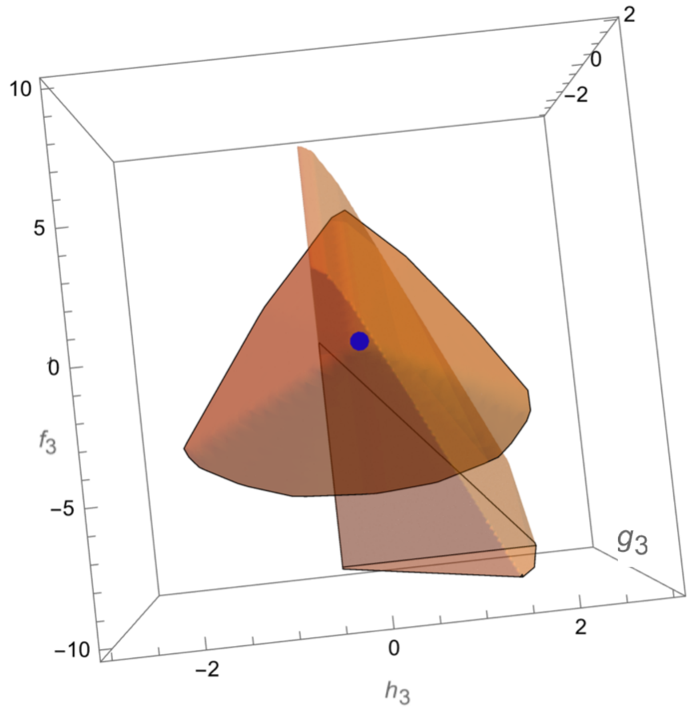


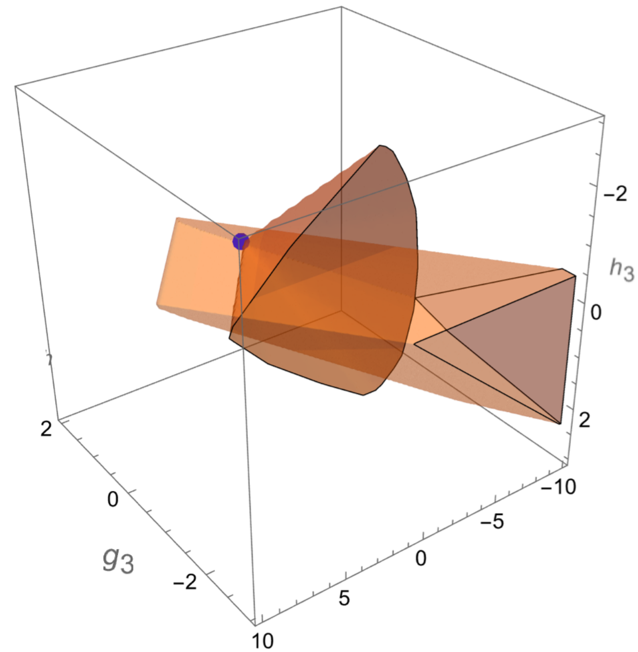
Causality vs positivity plots

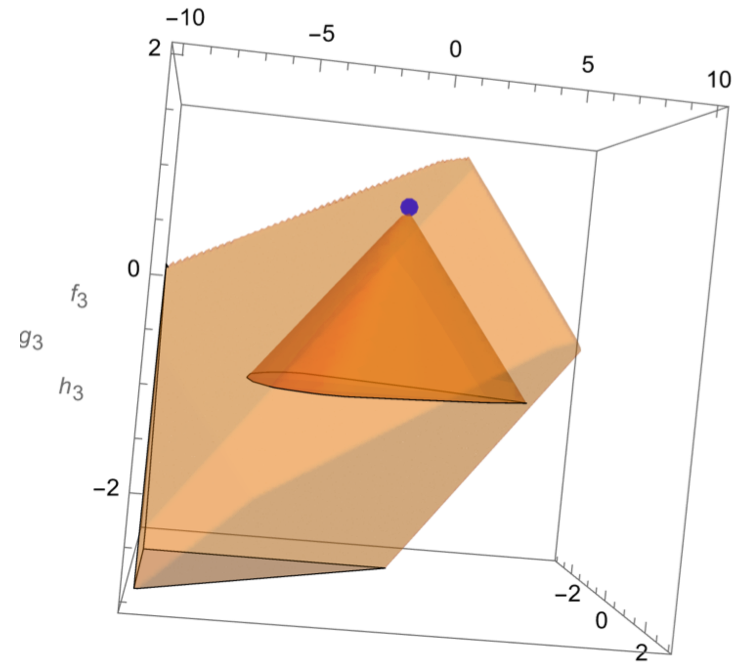


Causality vs positivity plots

$f_2 = f_4 = 0, \; g_4 = 1$





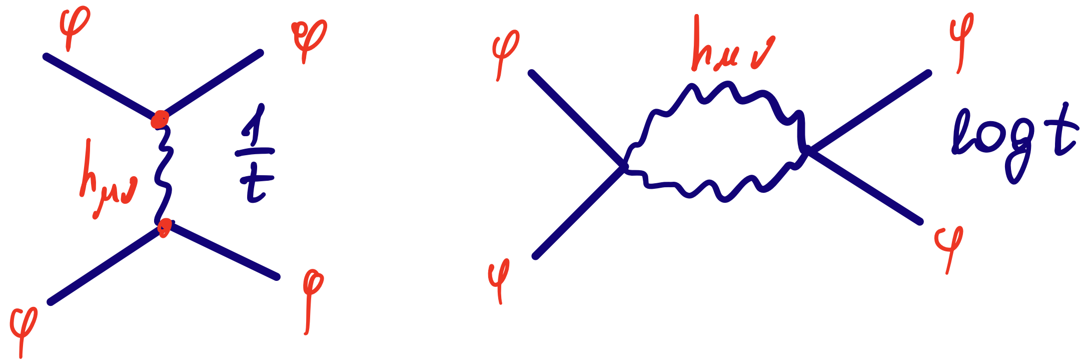


Summary of the photon bounds

- Indefinite helicity scalttering provides stronger bounds on EFT of photons. The optimal choice of polarization state may depend on the EFT couplings.
- Causality of the photon propagation is a condition independent of the assumptions about the UV completion - expected to be weaker than unitarity
- For $g_4 - f_4$ couplings positivity is stronger. Causality fails to give a compact bound.
- For $g_3 - h_3 - f_3$ couplings positivity and causality are complementary
- Some regions naively allowed by unitarity correspond to acausal propagation - positivity bounds can be improved further.

Dispersive relations with graviton exchange

Divergences at $t \to -0$



$$A(s, t \to -0) = A_0 \frac{s^2}{M_P^2 t} + A_1 \frac{s^2}{M_P^4} \log\left(\frac{-t}{\mu_0^2}\right) + \text{higher loops} + O(t)$$

$$\Sigma_0 = \frac{1}{2}\left(\frac{A_0}{t} + A_1 \log t\right) + (\text{loops}) + O(t)$$

Where are the same divergences in the right hand side?

How to cancel $1/t$ and $\log t$?

The only source of the divergences is an infinite tail of the integral.

$$\int_{M^2}^{\infty} \mathrm{Im}\mathcal{A}(s,t)ds\left(\frac{1}{s^3}+\frac{1}{(s+t)^3}\right)=f(t)+(\text{finite at } t\to 0)$$

Assume that after some scale M

$$\mathrm{Im}\mathcal{A}=s^{2+jt}\left(1+\frac{\xi}{\log s}\right)$$

J. Tokuda, K. Aoki, and S. Hirano, JHEP **11**, 054 (2020), arXiv:2007.15009 [hep-th].

This form allows to get $1/t$ and $\log t$ (Herrero-Valea, Santos-Garcia, AT'20). Generalisation:

$$\mathrm{Im}\mathcal{A}=s^{2+jt}\phi(s,t)$$

$$\phi(s,t)=\phi(s,0)+\phi_t(s,0)t+\frac{1}{2}\phi_{tt}(s,0)t^2+\dots \qquad s=M^2e^{\sigma}$$

$$\int_0^{\infty} 2\,\phi(s,0)s^{jt}\left(\frac{ds}{s}\right)=\int_0^{\infty} 2M^{2jt}\,\phi(\sigma,0)e^{j\sigma t}d\sigma=f(t)+(\text{finite at } t\to 0)$$

UV and IR are connected by the Laplace transformation

$$\phi(\sigma, 0) = \mathcal{L}^{-1}[f(t)] + O(t)$$

Next orders in t? Up to subleading terms in $t \to 0$ limit:

$$\phi(\sigma, 0) = a_0 L^{-1}[f(t)],$$

$$\phi_t(\sigma, 0) = a_1 L^{-1}\left[\frac{f(t)}{t}\right],$$

$$\phi_{tt}(\sigma, 0) = a_2 L^{-1}\left[\frac{f(t)}{t^2}\right], \quad \dots$$

$$f(t) = \frac{A}{t}, \quad \phi(\sigma, t) = \sum a_n \sigma^n t^n = \phi(\sigma t) = \phi(t \log s), \quad \phi(0) \neq 0$$

Recall that $A(s) < s^2$ at any $t \neq 0$. The dispersion relation allows to get the UV amplitude in the limit $t \log s \to 0$ while $t \to 0$ and $s \to \infty$.

Herrero-Valea, Koshelev, AT, arXiv:2205.13332

Reconstructing the amplitude from the imaginary part

$$\mathrm{Im} A(s,t) = i\gamma s^{2+jt}$$

$$\mathcal{A}(s,t) = \frac{s^2}{2\pi i}\oint_{\gamma_s}\frac{\mathcal{A}(z,t)dz}{z^2(z-s)} = F(s,t) + F(-s-t,t)$$

$$F(s,t) = \frac{s^2}{\pi}\int_0^\infty \frac{\mathrm{Im}\mathcal{A}(z,t)dz}{z^2(z-s)} = \frac{s^2}{\pi}\int_0^{M_*^2}\frac{\mathrm{Im}\mathcal{A}(z,t)dz}{z^2(z-s)} + $$

$$+\frac{s^2}{\pi}\int_{M_*^2}^\infty \frac{a_0 z^{jt}dz}{z-s} + \mathcal{O}\left(t\log(s)\right)$$

$$A(s,t) = -\frac{\gamma e^{-i\pi jt}}{\sin\left(\pi\alpha' t\right)}\left(s^{2+\alpha' t} + (-s-t)^{2+jt}\right) + \mathcal{O}\left(t\log(s)\right)$$

Real and imaginary parts are connected by analyticity!

Herrero-Valea, Koshelev, arXiv:2205.13332

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Conclusions

- UV and IR amplitudes are connected under the assumptions of unitarity, locality and analyticity of the fundamental theory
- Assumptions about UV lead to positivity bounds for IR theory
- Causality requirements does not lead to compact bounds while the positivity does. Causality doesn't rely on any UV properties (such as locality)
- However, causality bounds can improve the first EFT bounds coming from dispersive relations

With graviton exchange:

- IR singularities in the forward limit open the possibilities to find the form of UV amplitude in the limit $t \log s \to 0$, $s \to \infty$
- Gravity invalidates positivity bounds for $A''(s)$ but they still can be obtained from $A^{(4)}(s)$ and higher...

Thank you!

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UV completion	g_2	f_2	f_3	g_3	h_3	f_4	g_4	g'_4
scalar	1	1	3	1	0	$\frac{1}{2}$	1	0
axion	1	-1	-3	1	0	$-\frac{1}{2}$	1	0
scalar QED	1	$\frac{3}{4}$	$\frac{5}{14}$	$\frac{3}{28}$	$\frac{1}{28}$	$\frac{1}{84}$	$\frac{41}{420}$	$-\frac{1}{168}$
spinor QED	1	$-\frac{3}{11}$	$-\frac{10}{77}$	$\frac{4}{77}$	$-\frac{1}{77}$	$-\frac{1}{231}$	$\frac{13}{660}$	$-\frac{5}{462}$
vector QED	1	$\frac{1}{28}$	$\frac{5}{294}$	$-\frac{47}{1764}$	$\frac{1}{588}$	$\frac{1}{1764}$	$\frac{131}{8820}$	$-\frac{23}{1176}$
spin-2 even I*	1	1	0	1	0	$\frac{1}{2}$	1	-6
spin-2 even II	1	0	0	-1	0	0	1	-2
spin-2 odd*	1	-1	0	1	0	$-\frac{1}{2}$	1	-6
min.-coupled spin-2	1	0	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	-1

String amplitude

$$\mathcal{A}_{\text{string}}(s,t) = -A(s^2t^2 + s^2u^2 + t^2u^2)\, \frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}\bigg|_{u=-s-t}$$

J. H. Schwarz, Phys. Rept. **89**, 223 (1982).

$t \to 0, \;\; s \to \infty$

$$\mathcal{A}_{\text{string}}(s,t) = A\frac{s^2}{t}\left(1 + 2t\log(s) + 2t^2\log^2(s) + \frac{4}{3}t^3\log^3(s) + \frac{2}{3}t^4\log^4(s) + \dots\right)$$

Infinite arc contribution

Contribution from the arc R

$$\Sigma_\infty = -\frac{2\gamma e^{-i\pi jt}}{\sin(\pi jt)} \frac{R^{jt}}{2\pi} \frac{e^{2\pi ijt} - 1}{ijt} = -\frac{2\gamma}{\pi jt} R^{jt} = -\frac{2\gamma}{\pi jt}$$

Contribution from ImA

$$\Sigma_{UV} = \frac{2}{\pi} \int_{M_*^2}^{\infty} \frac{ds\, \mathrm{Im}\mathcal{A}(s,t)}{s^3} = \frac{2}{\pi} \int_{M_*^2}^{R} \frac{ds}{s} a_0 s^{jt} = \frac{2a_0}{\pi jt} \left(R^{jt} - (M_*^2)^{jt}\right)$$

Two limits can be considered: $R \to \infty$ with finite $t < 0$ and $t \log R \to 0$. The result is the same.