Gravitational reheating formulas in oscillating backgrounds

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Gravitational reheating is a mechanism to reheat the Universe (recall that, without reheating, inflation does not work), where **heavy** particles are produced when the inflaton field oscillates around the minimum of the potential, due to the interaction of a quantum field with the oscillating gravitational field (not through direct interaction with the inflaton).

Since, after inflation, the energy density of the produced particles (or that of their decay products) eventually has to dominate that of the inflaton for the Universe to become reheated, we demand that **the energy density of the inflaton decreases faster than that of radiation**. This requires that the EoS parameter satisfies $w_{eff} > 1/3$. For a potential behaving like φ^{2n}, this requires $n > 2$.

Otherwise, if the inflaton's energy density decreases more slowly than that of radiation, it could dominate again during this period, which is incompatible with the concordance model. Therefore, potentials appearing in **Starobinsky** ($n = 1$) or **Higgs** inflation, cannot include gravitational reheating as a mechanism to reheat the universe. (In that case one can use Instant Preheating or standard reheating based in quadratic interaction between the inflaton and the quantum field).

We begin by denoting $\rho_B(t)$ and $\langle\rho(t)\rangle$ as the energy densities of the background (the inflaton) and produced particles, respectively. Assuming that close to the minimum of the potential, which for simplicity we take as $\varphi_{min} = 0$, the potential behaves like φ^{2n}, for example:

$$V_n(\varphi) = \lambda M_{pl}^4 \left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{pl}}}\right)^{2n}, \tag{1}$$

we can conclude that, when the inflaton oscillates, the effective Equation of State (EoS) parameter is given by $w_{eff} = \frac{n-1}{n+1}$ (as a consequence of the "virial theorem"). Then, $n > 2 \Longrightarrow w_{eff} > 1/3$. During the oscillations of the inflaton, which takes place shortly after the end of inflation and prior to the decay of the produced particles, and denoting by "END" the end of inflation, the evolution of its energy density as a function of the **scale factor**, namely a, is:

$$\rho_B(t) = \rho_{B,END}\left(\frac{a_{END}}{a(t)}\right)^{3(1+w_{eff})} = \rho_{B,END}\left(\frac{a_{END}}{a(t)}\right)^{\frac{6n}{n+1}}. \tag{2}$$

Next we deal with the energy density of the produced particles with mass m_χ. Taking into account that its mean ingredient are the β**-Bogoliubov coefficient**, whose evolution, for the k-mode, is:

$$\begin{cases} \dot{\alpha}_k(t) = \frac{\dot{\omega}_k(t)}{2\omega_k(t)} e^{-2i\int^t \frac{\omega_k(t)}{a(t)}dt} \beta_k(t) \\ \dot{\beta}_k(t) = \frac{\dot{\omega}_k(t)}{2\omega_k(t)} e^{2i\int^t \frac{\omega_k(t)}{a(t)}dt} \alpha_k(t), \end{cases}, \quad \omega_k(t) = \sqrt{k^2 + m_\chi^2 a^2(t)}, \quad (3)$$

and the relationship $|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1$.
It is crucial to recognize that the β-Bogoliubov coeff. encapsulate both vacuum polarization effects and particle production. Shortly after the initiation of the oscillations, the polarization effects disappear. Thus, when they stabilize at a value β_k, the coeff. only reflect the contribution of the produced particles, and its energy density, is:

$$\langle\rho(t)\rangle = \frac{1}{2\pi^2 a^4(t)} \int_0^\infty k^2 \omega_k(t) |\beta_k|^2 dk \cong \frac{m_\chi}{2\pi^2 a^3(t)} \int_0^\infty k^2 |\beta_k|^2 dk. \quad (4)$$

Two distinct scenarios unfold: decay during and after the conclusion of the inflaton's domination. So, we will delve into both situations.

Decay before the end of the background domination

Let Γ be the decay rate of the heavy massive particles, and it is worth noting that **the decay process concludes when Γ is of the same order as the Hubble rate**, namely $H(t)$.

Denoting $\rho_{B,dec}$ and $\langle\rho_{dec}\rangle$ as the energy density of the background and that of the produced particles at the end of the decay, respectively, after the decay, they evolve as:

$$\rho_B(t) = \rho_{B,dec}\left(\frac{a_{dec}}{a(t)}\right)^{\frac{6n}{n+1}} \quad \text{and} \quad \langle\rho(t)\rangle = \langle\rho_{dec}\rangle\left(\frac{a_{dec}}{a(t)}\right)^{4}, \quad (5)$$

because, after the decay, the particles are currently relativistic.

Hence, given the virtually instantaneous nature of the thermalization process, **the universe undergoes reheating at the conclusion of the inflaton's domination**, denoted by the sub-index end, that is, when both energy densities are of the same order. This leads to the relation:

$$\left(\frac{a_{dec}}{a_{end}}\right)^{\frac{2(n-2)}{n+1}} = \frac{\langle\rho_{dec}\rangle}{\rho_{B,dec}} \Longrightarrow \langle\rho_{end}\rangle = \langle\rho_{dec}\rangle^{\frac{3n}{n-2}} \rho_{B,dec}^{-\frac{2(n+1)}{n-2}} . \quad (6)$$

Therefore, from the **Stefan-Boltzmann law**, the reheating temperature, for a potential like φ^{2n}, has the following expression:

$$T_{reh}(n) \equiv \left(\frac{30}{\pi^2 g_{reh}}\right)^{1/4} \langle\rho_{end}\rangle^{\frac{1}{4}} = \left(\frac{30}{\pi^2 g_{reh}}\right)^{1/4} \langle\rho_{dec}\rangle^{\frac{1}{4}} \left(\frac{\langle\rho_{dec}\rangle}{\rho_{B,dec}}\right)^{\frac{n+1}{2(n-2)}} , \quad (7)$$

where $g_{reh} = 106.75$ is the effective number of degrees of freedom for the Standard Model.
At this juncture, we can enhance this formula by considering the value of the corresponding energy densities at the decay. They follow the expressions:

$$\rho_{B,dec} = 3H_{END}^2 M_{pl}^2 \left(\frac{a_{END}}{a_{dec}}\right)^{\frac{6n}{n+1}} , \langle\rho_{dec}\rangle = \langle\rho_{END}\rangle \left(\frac{a_{END}}{a_{dec}}\right)^3 . \quad (8)$$

Then, when the heavy particles have completely decayed, which occurs when $H \sim \Gamma$, **the semi-classical Friedmann equation** $H^2 = \frac{1}{3M_{pl}^2}(\rho_B + \langle\rho\rangle)$, becomes:

$$3\Gamma^2 M_{pl}^2 = \rho_{B,END} x^{\frac{2n}{n+1}} + \langle\rho_{END}\rangle x \Longrightarrow \frac{3\Gamma^2 M_{pl}^2}{\rho_{B,END}} = x^{\frac{2n}{n+1}} + \frac{\langle\rho_{END}\rangle}{\rho_{B,END}} x, \quad (9)$$

where $x = \left(\frac{a_{END}}{a_{dec}}\right)^3$. The solution has the form

$$x = F_n(\Theta_1, \Theta_2), \quad (10)$$

where F_n is a function which depend on n and the parameters: $\Theta_1 = \frac{\langle\rho_{END}\rangle}{\rho_{B,END}}$ and $\Theta_2 = \frac{3\Gamma^2 M_{pl}^2}{\rho_{B,END}}$. For example, when $n = \infty$, i.e. when $w_{eff} = 1$, ones has:

$$F_\infty(\Theta_1, \Theta_2) = \frac{1}{2}\left(\sqrt{\Theta_1^2 + 4\Theta_2} - \Theta_1\right), \quad (11)$$

and for $n=3$ we obtain $x^{3/2}+\Theta_1 x-\Theta_2=0$ which is a cubic equation when one introduces the variable $x=z^2$. This equation can be solved using the **Cardano's formulas**, obtaining:

$$F_3(\Theta_1,\Theta_2)=\left[\sqrt[3]{\frac{1}{2}\left(\Theta_2-\frac{2\Theta_1^3}{27}+\sqrt{\Theta_2\left(\Theta_2-\frac{4\Theta_1^3}{27}\right)}\right)}+\sqrt[3]{\frac{1}{2}\left(\Theta_2-\frac{2\Theta_1^3}{27}-\sqrt{\Theta_2\left(\Theta_2-\frac{4\Theta_1^3}{27}\right)}\right)}-\frac{\Theta_1}{3}\right]^2, \qquad (12)$$

provided that $\Theta_2-\frac{4\Theta_1^3}{27}>0$, what always happens.
Effectively, from the equation $x^{3/2}+\Theta_1 x-\Theta_2=0$ we have the bound $\Theta_2>\Theta_1 x$. On the other hand, since the decay occurs during the domination of the inflaton's energy density, one has $\langle\rho_{dec}\rangle\leq\rho_{B,dec}$ which is equivalent to $\Theta_1^2\leq x$, then we have:

$$\Theta_2-\frac{4\Theta_1^3}{27}>\Theta_1 x\left(1-\frac{4}{27}\right)>0. \qquad (13)$$

Coming back to the reheating formula, we can write:

$$T_{reh}(n) = \left(\frac{90}{\pi^2 g_{reh}}\right)^{1/4} \left(\frac{\Theta_1^3}{F_n(\Theta_1, \Theta_2)}\right)^{\frac{n}{4(n-2)}} \sqrt{H_{END} M_{pl}}. \qquad (14)$$

Since the decay occurs during the domination of the inflaton's energy density, we have the constraints $\Gamma \leq H_{END}$ and $\langle \rho_{dec} \rangle \leq \rho_{B,dec}$, which after some algebra, is equivalent to $\Theta_1^{\frac{n+1}{n-1}} \leq F_n(\Theta_1, \Theta_2)$. Finally, we calculate the **maximum reheating temperature**, which is obtained when $\langle \rho_{dec} \rangle = \rho_{B,dec}$, that is, when $F_n(\Theta_1, \Theta_2) = \Theta_1^{\frac{n+1}{n-1}}$, and thus, the maximum reheating temperature is given by:

$$T_{reh}^{max}(n) \cong 5 \times 10^{-1} \Theta_1^{\frac{n}{2(n-1)}} \sqrt{H_{END} M_{pl}}, \quad \Theta_1 = \frac{\langle \rho_{END} \rangle}{\rho_{B,END}}. \qquad (15)$$

Suppose that the particles decay before their energy dominates that of the inflaton. Then, from this moment, their energy will decrease faster than if the particles had not decayed, and thus, the energy of the particles will take longer to surpass that of the inflaton. Therefore, the reheating temperature will be lower.

Decay after the end of the background domination

We will examine the scenario where the decay occurs after the end of domination of the inflation's energy density. Given the instantaneous nature of the thermalization process, reheating is concluded upon the completion of decay. Therefore, the reheating temperature is:

$$T_{reh} = \left(\frac{30}{\pi^2 g_{reh}} \right)^{1/4} \langle \rho_{dec} \rangle^{1/4}, \tag{16}$$

where one must ensure that $\Gamma \leq H_{end}$. The value of the Hubble rate at the end of the background domination can be calculated by considering that, in this scenario, the energy density of the produced particles decays as a^{-3} throughout the entire domination of the inflaton's energy density. Thus, at the end of the background domination:

$$\Theta_1 = \frac{\langle \rho_{END} \rangle}{\rho_{B,END}} = \left(\frac{a_{END}}{a_{end}} \right)^{\frac{3(n-1)}{n+1}} \Longrightarrow H_{end}^2 = 2H_{END}^2 \Theta_1^{\frac{2n}{n-1}}, \tag{17}$$

and thus, the constraint $\Gamma \leq H_{end}$, becomes $\Gamma \leq \sqrt{2}\Theta_1^{\frac{n}{n-1}} H_{END}$.

To refine the formula for the reheating temperature (16), we perform the following calculation:

$$\langle\rho_{end}\rangle = \rho_{B,END}\Theta_1^{\frac{2n}{n-1}} \Longrightarrow \langle\rho_{dec}\rangle^{1/4} = \rho_{B,END}^{1/4}\Theta_1^{\frac{n}{2(n-1)}}\left(\frac{a_{end}}{a_{dec}}\right)^{3/4}. \quad (18)$$

Therefore, when the decay is immediately finished, introducing the notation $y = \left(\frac{a_{end}}{a_{dec}}\right)^3$, the semi-classical Friedmann is given by:

$$3\Gamma^2 M_{pl}^2 = \rho_{B,end}(y + y^{\frac{2n}{n+1}}) \Longrightarrow y^{\frac{2n}{n+1}} + y - \Theta_2\Theta_1^{-\frac{2n}{n-1}} = 0, \quad (19)$$

where we have used that $\frac{3\Gamma^2 M_{pl}^2}{\rho_{B,end}} = \Theta_2\Theta_1^{-\frac{2n}{n-1}}$.

Writing the solution as:

$$\left(\frac{a_{end}}{a_{dec}}\right)^3 = G_n(\Theta_2\Theta_1^{-\frac{2n}{n-1}}), \quad (20)$$

and thus, using the formula (18), the reheating temperature takes the following form:

$$T_{reh}(n) = \left(\frac{90G_n(\Theta_2\Theta_1^{-\frac{2n}{n-1}})}{\pi^2 g_{reh}}\right)^{1/4} \Theta_1^{\frac{n}{2(n-1)}} \sqrt{H_{END}M_{pl}}. \quad (21)$$

Finally, we can see the simplest expression of this reheating temperature is obtained when $n = \infty$, obtaining

$$G_\infty(\Theta_2\Theta_1^{-2}) = \frac{1}{2\Theta_1^2}\left(\sqrt{\Theta_1^4 + 4\Theta_2^2} - \Theta_1^2\right), \quad (22)$$

and thus,

$$T_{reh}(\infty) = \left(\frac{45(\sqrt{\Theta_1^4 + 4\Theta_2^2} - \Theta_1^2)}{\pi^2 g_{reh}}\right)^{1/4} \sqrt{H_{END}M_{pl}}. \quad (23)$$

Important: To have a succesfull BBN we demand $1\text{MeV} < T_{reh} < 10^9\text{GeV}$. Note that 10^9 GeV is a conservative bound.
To obtain the maximum reheating temperature we have to calculate the value of Θ_1. The problem is that $\langle \rho_{END} \rangle$ has to be calculated numerically, and in the literature only appears analytic formulas in the quadratic case $V(\varphi) = \frac{1}{2}m_\varphi^2\varphi^2$. However for a quadratic model, $w_{eff} = 0$, that is, the energy density of the inflaton scales as matter, what **makes impossible a successful reheating**.
Anyway, to have an analytic formula for the maximum reheating temperature, we will assume that for values of n of the order 1, the energy density of the produced particles is of the same order than the one obtained in the quadratic case. Therefore, we will start using the formula (Y. Ema, K. Nakayama and Y. Tang, JHEP **09** 135 (2018)):

$$\langle \rho_{END} \rangle \cong C H_*^3 m_\chi \left(\frac{m_\chi}{m_\varphi} \right)^4, \tag{24}$$

where $m_\chi \ll m_\varphi$, H_* is the scale of inflation, i.e., the value of the Hubble rate at the horizon crossing and $C \cong 2 \times 10^{-3}$ is a dimensionless constant.

Firstly, we have to take into account that for a quadratic potential one has

$$\epsilon_* = \frac{1}{4}(1-n_s), \qquad \varphi_*^2 = \frac{8M_{pl}^2}{1-n_s}, \tag{25}$$

where n_s is **the spectral index**,
Next, we will use that, at the horizon crossing, the Friedmann equation is:

$$H_*^2 \cong \frac{1}{6M_{pl}^2} m_\varphi^2 \varphi_*^2, \tag{26}$$

and from the *formula of the power spectrum of scalar perturbations* $\frac{H_*^2}{8\pi^2\epsilon_* M_{pl}^2} \sim 2\times 10^{-9}$ we get:

$$m_\varphi^2 \sim 3\pi^2 \times 10^{-9}(1-n_s)^2 M_{pl}^2 \cong 3\pi^2 \times 10^{-12} M_{pl}^2, \tag{27}$$

and

$$H_* \cong 2\pi\sqrt{10(1-n_s)}10^{-5} M_{pl} \sim 4\times 10^{-5} M_{pl}, \tag{28}$$

where we have taken $n_s \cong 0.96$.

Inserting the value of m_φ in the formula (24) one finds:

$$\langle \rho_{END} \rangle \sim 2 \times 10^{18} H_*^3 m_\chi \left(\frac{m_\chi}{M_{pl}} \right)^4, \tag{29}$$

for $m_\chi \ll m_\varphi \sim 5 \times 10^{-6} M_{pl}$. Therefore, we will obtain:

$$\Theta_1 \sim 6 \times 10^{17} \frac{H_*^3 m_\chi}{H_{END}^2 M_{pl}^2} \left(\frac{m_\chi}{M_{pl}} \right)^4. \tag{30}$$

To find a bound for the maximum reheating temperature, note that, for a quadratic model $H_{END} = \frac{1}{\sqrt{2}} m_\varphi \sim 4 \times 10^{-6} M_{pl}$ and, as we have already seen in last slide, $H_* \sim 4 \times 10^{-5} M_{pl}$. Then, one gets $\Theta_1 \cong 2 \times 10^{15} \left(\frac{m_\chi}{M_{pl}} \right)^5$. Therefore, taking into account that $T_{reh}^{max}(n) \cong 5 \times 10^{-1} \Theta_1^{\frac{n}{2(n-1)}} \sqrt{H_{END} M_{pl}}$, and the bound $\Theta_1^{\frac{n}{2(n-1)}} < \sqrt{\Theta_1}$ for $n \geq 2$, one gets:

$$T_{reh}^{max}(n) < 5 \times 10^4 \left(\frac{m_\chi}{M_{pl}} \right)^{5/2} M_{pl} < 10^{-13} M_{pl} \sim 2 \times 10^5 \text{GeV}. \tag{31}$$

Next, we will consider another analytic formula for the energy density of the produced particles (E. W. Kolb and A. J. Long, (2024) [arXiv:2312.09042]):

$$\langle \rho_{END} \rangle = \frac{H_{END}^2 m_\chi^2}{16\pi^2}, \tag{32}$$

for conformally coupled particles satisfying $m_\chi < H_{END}$.
In this case,

$$\Theta_1 = \frac{m_\chi^2}{48\pi^2 M_{pl}^2} \sim 2 \times 10^{-3} \frac{m_\chi^2}{M_{pl}^2}. \tag{33}$$

Finally, we have:

$$T_{reh}^{max}(n) < 4 \times 10^{-5} m_\chi < 4 \times 10^{-12} M_{pl} \sim 10^7 \text{GeV}, \tag{34}$$

where we have taken $H_{END} \sim 4 \times 10^{-6} M_{pl}$ and $m_\chi < 10^{-7} M_{pl}$.

MORE ACCURATE BOUNDS:

Case $n = 3$.

$$V_3(\varphi) = \lambda M_{pl}^4 \left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{pl}}}\right)^6 . \qquad (35)$$

For this potential, we will have $w_{eff} = 1/2$, $\lambda \sim 5 \times 10^{-11}$, $H_{END} \sim 9 \times 10^{-7} M_{pl}$ and $H_* \cong \sqrt{\frac{\lambda}{3}} M_{pl} \cong 4 \times 10^{-6} M_{pl}$.
In this case, the maximum reheating temperature becomes:

$$T_{reh}^{max}(3) \cong 5 \times 10^{-4} \Theta_1^{3/4} M_{pl} \cong 2 \times 10^5 \frac{\langle \rho_{END} \rangle^{3/4}}{M_{pl}^2}, \qquad (36)$$

where we have used that $\Theta_1 \cong 4 \times 10^{11} \frac{\langle \rho_{END} \rangle}{M_{pl}^4}$.
If one considers, the gravitational particle production given by the formula (29), i.e., $\langle \rho_{END} \rangle \sim 2 \times 10^{18} H_*^3 m_\chi \left(\frac{m_\chi}{M_{pl}}\right)^4$, one gets:

$$T_{reh}^{max}(3) \sim 8 \times 10^6 \left(\frac{m_\chi}{M_{pl}}\right)^3 (m_\chi^3 M_{pl})^{1/4} < 5 \times 10^{-20} M_{pl} \sim 10^2 \text{MeV}, \qquad (37)$$

for $m_\chi < 10^{-7} M_{pl}$.

On the other hand, to have a successful reheating we have to impose $T_{reh} > 1$ MeV, what constraints the mass to the range:

$$3 \times 10^{-8} M_{pl} < m_\chi < 10^{-7} M_{pl}. \tag{38}$$

And considering (32), i.e., $\langle \rho_{END} \rangle = \frac{H_{END}^2 m_\chi^2}{16\pi^2}$, for $n = 3$, we will have:

$$T_{reh}^{max}(3) \sim 3 \times 10^{-6} \left(\frac{m_\chi}{M_{pl}} \right)^{3/2} M_{pl} < 9 \times 10^{-17} M_{pl} \sim 2 \times 10^2 \text{GeV}. \tag{39}$$

In that case, the viable masses must satisfy:

$$3 \times 10^{-11} M_{pl} < m_\chi < 10^{-7} M_{pl}. \tag{40}$$

Note also that, making the same kind of calculation, for not viable potentials, i.e., as we have already discussed, for $n \leq 2$, the maximum reheating temperature is less than 10 MeV.

Case $n \to \infty$. We consider the extreme case, where after inflation the universe enters in a stiff or kination phase ($w_{eff} = 1$).
In this case the maximum reheating temperature is given by $T_{reh}^{max}(\infty) \cong 5 \times 10^{-1}\sqrt{\Theta_1 H_{END} M_{pl}}$, and the expression of the parameter Θ_1 is:

$$\Theta_1 = \frac{\langle \rho_{END} \rangle}{3H_{END}^2 M_{pl}^2} \sim 3 \times 10^{11} \frac{\langle \rho_{END} \rangle}{M_{pl}^4}, \qquad (41)$$

where we have taken $H_{END} \sim 10^{-6} M_{pl}$.
Therefore, recalling that from the analytic formula (24), we have obtained $\langle \rho_{END} \rangle \sim 2 \times 10^{18} H_*^3 m_\chi \left(\frac{m_\chi}{M_{pl}}\right)^4$, we will have:

$$\Theta_1 \sim 6 \times 10^{29} \frac{H_*^3 m_\chi}{M_{pl}^4} \left(\frac{m_\chi}{M_{pl}}\right)^4 \sim 4 \times 10^{13} \left(\frac{m_\chi}{M_{pl}}\right)^5, \qquad (42)$$

where we have chosen $H_* \sim 4 \times 10^{-6} M_{pl}$. And the maximum reheating temperature becomes:

$$T_{reh}^{max}(\infty) \sim 3 \times 10^3 \left(\frac{m_\chi}{M_{pl}}\right)^{5/2} M_{pl} < 10^{-14} M_{pl} \sim 2 \times 10^4 \text{GeV}, \quad (43)$$

where we have used that $m_\chi < 10^{-7} M_{pl} \sim 2 \times 10^{11}$ GeV.
In that case the viable masses are the ones satisfying:

$$6 \times 10^{-11} M_{pl} < m_\chi < 10^{-7} M_{pl}. \quad (44)$$

And for the analytic formula (32), i.e., $\langle \rho_{END} \rangle = \frac{H_{END}^2 m_\chi^2}{16\pi^2}$, we have already seen that $\Theta_1 \sim 2 \times 10^{-3} \frac{m_\chi^2}{M_{pl}^2}$, obtaining:

$$T_{reh}^{max}(\infty) \sim 2 \times 10^{-5} \frac{m_\chi}{M_{pl}} M_{pl} < 2 \times 10^{-12} M_{pl} \sim 5 \times 10^6 \text{GeV}. \quad (45)$$

And the viable masses are in the range:

$$10^{-17} M_{pl} < m_\chi < 10^{-7} M_{pl}. \quad (46)$$

Born approximation

In the Born approximation, the creation rate of χ-particles, satisfying $m_\chi < \omega(t)$, is:

$$\Gamma_\chi(t) \sim \frac{m_\chi^4}{16\pi(n+1)^2\omega(t)M_{pl}^4}\varphi^2(t) \quad \text{with} \quad V_n(\varphi) \sim \varphi^{2n}, \tag{47}$$

where the leading frequency ω is approximately

$$\omega(t) \sim \frac{\sqrt{\rho_B(t)}}{\varphi(t)} \Longrightarrow \omega_{END} \sim \frac{\sqrt{3}H_{END}M_{pl}}{\varphi_{END}} \tag{48}$$

Next, we use the evolution equation:

$$\frac{d\langle\rho(t)\rangle}{dt} = \Gamma_\chi(t)\rho_B(t), \tag{49}$$

which after one Hubble time can be approximated by

$$\langle\rho_{END}\rangle \cong \frac{\Gamma_{\chi,END}}{H_{END}}\rho_{B,END}, \tag{50}$$

and thus,

$$\Theta_1 = \frac{\langle \rho_{END} \rangle}{\rho_{B,END}} \cong \frac{\Gamma_{\chi,END}}{H_{END}} \cong \frac{1}{16\sqrt{3}\pi(n+1)^2} \left(\frac{m_\chi}{M_{pl}} \right)^4 \frac{\varphi_{END}^3}{H_{END}^2 M_{pl}}. \quad (51)$$

On the other hand, to obtain φ_{END} and H_{END} we solve $\epsilon \equiv \frac{M_{pl}^2}{2} \left(\frac{V_n'}{V_n} \right)^2 = 1$ for $V_n(\varphi) = \lambda M_{pl}^4 \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\varphi}{M_{pl}}} \right)^{2n}$, obtaining:

$$\varphi_{END} = -\sqrt{\frac{3}{2}} \ln \left(\frac{2\sqrt{3}n - 3}{4n^2 - 3} \right) M_{pl} \quad (52)$$

$$H_{END} = \sqrt{\frac{\lambda}{2}} \left(1 - \frac{2\sqrt{3}n - 3}{4n^2 - 3} \right)^n M_{pl}. \quad (53)$$

And from the formula of the power spectrum of scalar perturbation $\frac{H_*^2}{8\pi^2 \epsilon_* M_{pl}^2} \sim 2 \times 10^{-9}$ and $3M_{pl}^2 H_*^2 \sim V_n(\varphi_*) \sim \lambda M_{pl}^4$, we get:

$$\lambda \sim 3\pi^2 (1 - n_s)^2 10^{-9} \cong 5 \times 10^{-11} \quad \text{where} \quad n_s \cong 0.96. \quad (54)$$

Therefore, we arrive at:

$$\Theta_1 \cong 10^9 \frac{1}{(n+1)^2} \left(\frac{m_\chi}{M_{pl}} \right)^4 \ln^3 \left(\frac{4n^2-3}{2\sqrt{3}n-3} \right) \left(1 - \frac{2\sqrt{3}n-3}{4n^2-3} \right)^{-2n} . \quad (55)$$

and the maximum reheating temperature is given by:

$$T_{reh}^{max}(n) \cong 4 \times 10^{\frac{2n+7}{2(n-1)}} \frac{1}{(1+n)^{\frac{n}{n-1}}} \left(\frac{m_\chi}{M_{pl}} \right)^{\frac{2n}{n-1}} \left(\ln \left(\frac{4n^2-3}{2\sqrt{3}n-3} \right) \right)^{\frac{3n}{2(n-1)}} \left(1 - \frac{2\sqrt{3}n-3}{4n^2-3} \right)^{-\frac{n(n+1)}{n-1}} M_{pl} . \quad (56)$$

	T_{reh}^{max} for $m_\chi < 2\times 10^{11}$ GeV	Viable masses
$n=3$	$10^4 \left(\frac{m_\chi}{M_{pl}}\right)^3 M_{pl} < 20$GeV	10^{10} GeV $< m_\chi$
$n=4$	$2\times 10^3 \left(\frac{m_\chi}{M_{pl}}\right)^{8/3} M_{pl} < 10^2$GeV	10^9 GeV $< m_\chi$
$n=5$	$6\times 10^2 \left(\frac{m_\chi}{M_{pl}}\right)^{5/2} M_{pl} < 5\times 10^3$GeV	5×10^8 GeV $< m_\chi$
$n=6$	$4\times 10^2 \left(\frac{m_\chi}{M_{pl}}\right)^{12/5} M_{pl} < 10^4$GeV	2×10^8 GeV $< m_\chi$

Application to α-attractors

We deal with α-attractors whose potential is:

$$V_n(\varphi) = \lambda M_{pl} \left(\sqrt{6} \tanh \left(\frac{\varphi}{\sqrt{6} M_{pl}} \right) \right)^{2n}, \quad (57)$$

with (from the formula of the power spectrum of scalar perturbations):

$$\lambda \cong \frac{36\pi^2}{6^n N_*^2} 10^{-9} \cong 6^{-n} 10^{-10}, \quad (58)$$

where we have chosen as the number of last efolds $N_* = 55$.
In the minimally coupled case, the authors of arXiv:2404.06530 numerically found that for masses of the order $m_\chi \sim 10^{-4} H_{END}$, the energy density of the *minimally coupled* produced particles, when $n = 4$, is:

$$\langle \rho_{END} \rangle \cong 10^2 H_{END}^3 m_\chi = 10^{-2} H_{END}^4, \quad (59)$$

what leads to:

$$\Theta_1 \cong 3 \times 10^{-3} \left(\frac{H_{END}}{M_{pl}} \right)^2 . \qquad (60)$$

Consequently, the maximum reheating temperature is:

$$T_{reh}^{max}(4) = \left(\frac{90}{\pi^2 g_{reh}} \right)^{1/4} \Theta_1^{2/3} \sqrt{H_{END} M_{pl}} \cong 10^{-2} \left(\frac{H_{END}}{M_{pl}} \right)^{11/6} M_{pl} . \qquad (61)$$

To find H_{END}, we solve the equation $\frac{M_{pl}^2}{2}\left(\frac{V_n'}{V_n}\right)^2 = 1$, obtaining:

$$\sinh^2 \left(\frac{2\varphi_{END}}{\sqrt{6} M_{pl}} \right) = \frac{2n^2}{9}, \qquad (62)$$

which after some algebra, one gets:

$$V_n(\varphi_{END}) \cong 10^{-10} M_{pl}^4 \left(\frac{18 + 2n^2 - 6\sqrt{9 + 2n^2}}{2n^2} \right)^n , \qquad (63)$$

and thus:

$$H_{END} \cong 7 \times 10^{-6} \left(\frac{18 + 2n^2 - 6\sqrt{9 + 2n^2}}{2n^2} \right)^{n/2} M_{pl}. \tag{64}$$

Then, for $n = 4$ one has $H_{END} \sim 10^{-6} M_{pl}$, what leads to the following maximum reheating temperature:

$$T_{reh}^{max}(4) \cong 10^{-13} M_{pl} \cong 2 \times 10^5 \text{ GeV}. \tag{65}$$

In the same way, for $n = 8$, one gets:

$$T_{reh}^{max}(8) \cong 8 \times 10^{-13} M_{pl} \cong 2 \times 10^6 \text{ GeV}. \tag{66}$$

Numerical calculations

Potential: $V_3(\varphi) = 5 \times 10^{-11} M_{\rm pl}^4 \left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm pl}}}\right)^6$

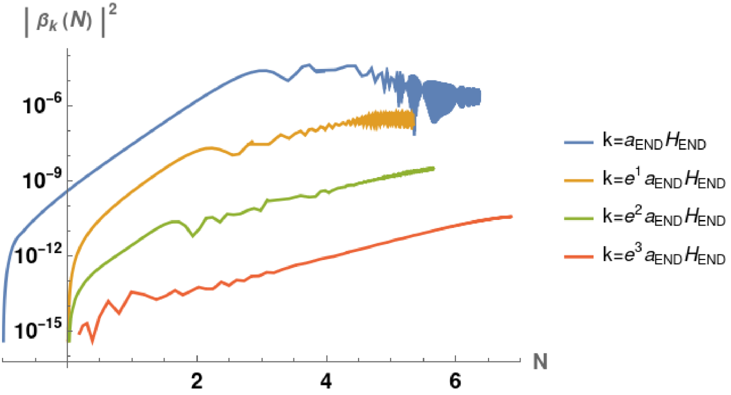

Figure: Case $n=3$: Stabilisation of the value of $|\beta_k|^2$ after the end of inflation for $m_\chi = 10^{-2} H_{END}$ in function of the number of e-folds after the end of inflation for different modes.

We verified that the value of $|\beta_k|^2$ attains a non-trivial value for modes $k < e^5 a_{END} H_{END}$ and the contribution of the modes satisfying $k \leq e^{-3} a_{END} H_{END}$ to the value of $\langle \rho_{END} \rangle$ is negligible. Once we have obtained the values of the β-Bogoliubov coefficients we approximate the value of $\langle \rho_{END} \rangle$ via a Riemann sum as:

$$\langle \rho_{\rm END} \rangle \cong 3 \times 10^{-3} H^2_{\rm END} m^2_\chi. \tag{67}$$

Therefore,

$$\Theta_1 \cong 10^{-3} \left(\frac{m_\chi}{M_{\rm pl}} \right)^2 = 10^{-7} \left(\frac{H_{END}}{M_{\rm pl}} \right)^2, \tag{68}$$

and the maximum reheating temperature, for masses of the order $m_\chi \sim 10^{-2} H_{END}$, is given by:

$$T^{\rm max}_{\rm reh}(3) \cong 3 \times 10^{-6} \frac{H^2_{\rm END}}{M_{\rm pl}} \cong 2 \times 10^{-18} M_{\rm pl} \sim 5 \text{ GeV}. \tag{69}$$

CONCLUSIONS:

1. For potentials that near the minimum behave like φ^{2n}, the maximum reheating temperature is:

$$T_{reh}^{max}(n) \cong 5 \times 10^{-1} \Theta_1^{\frac{n}{2(n-1)}} \sqrt{H_{END} M_{pl}}, \quad (70)$$

where $\Theta_1 = \frac{\langle \rho_{END} \rangle}{3H_{END}^2 M_{pl}^2}$. Note that the value of the Hubble rate at the end of inflation can be calculated analytically.

2. For the analytic formulas, **obtained for no viable quadratic potentials**, with $m_\chi < 10^{-7} M_{pl}$:

$$\langle \rho_{END} \rangle = \frac{H_{END}^2 m_\chi^2}{16\pi^2}, \quad \langle \rho_{END} \rangle \sim 2 \times 10^{18} H_*^3 m_\chi \left(\frac{m_\chi}{M_{pl}} \right)^4, \quad (71)$$

the maximum reheating temperature is bounded by $1\text{MeV} < T_{reh}^{max} < 10^9\text{GeV}$, overpassing the gravitino problem and obtaining a reheated universe before the BBN.

3. We have seen that **the maximum reheating temperature increases** as n increases.
4. We have found the values of the mass m_χ that yield a viable maximum reheating temperature in the range from 1 MeV to 10^9 GeV.
   In all situations $n = 3, 4, ...$, masses satisfying the constraint:

$$3 \times 10^{-8} M_{pl} < m_\chi < 10^{-7} M_{pl}, \tag{72}$$

   lead to a viable maximum reheating temperature.
   Of course, when n increases the range of the viable masses is larger.

Thank you for your attention.