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Canonical construction of invariant differential operators

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Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory (for reviews, cf. e.g., Maldacena,Terning]. Thus, it is important for the applications in physics to study systematically such operators. For more relevant references cf., e.g., [VKD1].

Recently we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study. One first choice would be non-compact groups that have discrete series of representations. By the Harish-Chandra criterion these are groups where holds:

$$\text{rank}\, G = \text{rank}\ K,$$

where K is the maximal compact subgroup of the non-compact group G. Another formulation is to say that the Lie algebra $\mathcal{G}$ of G has a compact Cartan subalgebra.

Example: the groups $SO(p,q)$ have discrete series, except when both p,q are odd numbers.

This class is rather big, thus, we decided to consider a subclass, namely, the class of Hermitian symmetric spaces. The practical criterion is that in these cases, the maximal compact subalgebra $\mathcal{K}$ is of the form:

$$\mathcal{K} \;=\; so(2) \oplus \mathcal{K}'$$

The Lie algebras from this class are:

$$so(n,2), \quad sp(n,R), \quad su(m,n), \quad so^*(2n),$$
$$E_{6(-14)}, \quad E_{7(-25)}$$

These groups/algebras have highest/lowest weight representations, and relatedly *holomorphic* discrete series representations.

The most widely used of these algebras are the conformal algebras so(n,2) in n-dimensional Minkowski space-time. In that case, there is a maximal Bruhat decomposition with direct physical meaning:

$$so(n,2) \;=\; \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}} \;,$$
$$\mathcal{M} \;=\; so(n-1,1) \;, \quad \dim \mathcal{A} = 1,$$
$$\dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n$$

where $so(n-1,1)$ is the Lorentz algebra of n-dimensional Minkowski space-time, the subalgebra $\mathcal{A} = so(1,1)$ represents the dilatations, the conjugated subalgebras $\mathcal{N}$, $\tilde{\mathcal{N}}$ are the algebras of translations, and special conformal transformations, both being isomorphic to n-dimensional Minkowski space-time.

The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ ($\cong \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$) is a maximal parabolic subalgebra.

There are other special features which are important. In particular, the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\begin{aligned} \mathcal{K}^{\mathbb{C}} &= so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) \cong \\ &\cong so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}} \ . \end{aligned}$$

In particular, the coincidence of the complexification of the semi-simple subalgebras:

$$\mathcal{K}'^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \qquad (*)$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $\mathcal{K}'$. The latter leads to the fact that the corresponding induced representations are representations of finite $\mathcal{K}$-type [Harish-Chandra].

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of the conformal algebra $so(n,2)$. This subclass consists of:

$so(n,2), \quad sp(n,\mathbb{R}), \quad su(n,n), \quad so^*(4n), \quad E_{7(-25)}$

the corresponding analogs of Minkowski space-time V being:

$$\mathbb{R}^{n-1,1}, \quad \mathrm{Sym}(n,\mathbb{R}), \quad \mathrm{Herm}(n,\mathbb{C}),$$
$$\mathrm{Herm}(n,\mathbb{Q}), \quad \mathrm{Herm}(3,\mathbb{O})$$

In view of applications to physics, we proposed to call these algebras 'conformal Lie algebras', (or groups).

The corresponding groups are also called '*Hermitian symmetric spaces of tube type*' [Faraut-Koranyi]. The same class was identified from different considerations in [Gunaydin] called there '*conformal groups of simple Jordan algebras*'. In fact, the relation between Jordan algebras and division algebras was known long time ago. Our class was identified from still different considerations also in [Mack–de-Riese] where they were called '*simple space-time symmetries generalizing conformal symmetry*'.

We have started the study of the above class in the framework of the present approach in the cases: $so(n,2)$, $su(n,n)$, $sp(n,\mathbb{R})$, $E_{7(-25)}$, we have considered also the algebra $E_{6(-14)}$, and others for which I would have no time today.

Lately, we discovered an efficient way to extend our considerations beyond this class introducing the notion of 'parabolically related non-compact semisimple Lie algebras'.

• *Definition:* Let $\mathcal{G}, \mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them parabolically related if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ $(\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}})$.$\diamondsuit$

Certainly, there are many such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two fixed algebras $\mathcal{G}, \mathcal{G}'$ may be parabolically related via different parabolic subalgebras.

We summarize the algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling $(*)$ in the following table:

Table of conformal Lie algebras (CLA) $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling $(*)$
and the corresponding parabolically related algebras $\mathcal{G}'$,
also some non-CLA cases

$\mathcal{G}$	$\mathcal{K}$	$\mathcal{M}$ $\dim V$	$\mathcal{G}'$	$\mathcal{M}'$
$so(n,2)$ $n \geq 3$	$so(n) \oplus so(2)$	$so(n-1,1)$ n	$so(p,q)$, $p+q=$ $=n+2$	$so(p-1,q-1)$
$su(n,n)$ $n \geq 3$	$u(n) \oplus su(n)$	$sl(n,\mathbb{C})_{\mathbb{R}}$ n^2	$sl(2n,\mathbb{R})$ $su^*(2n)$, $n=2k$	$sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$ $su^*(2k) \oplus su^*(2k)$
$sp(n,\mathbb{R})$ rank $= n \geq 3$	$u(n)$	$sl(n,\mathbb{R})$ $n(n+1)/2$	$sp(r,r)$, $n=2r$	$su^*(2r)$, $n=2r$
$so^*(4n)$ $n \geq 3$	$u(2n)$	$su^*(2n)$ $n(2n-1)$	$so(2n,2n)$	$sl(2n,\mathbb{R})$
$E_{7(-25)}$	$e_6 \oplus so(2)$	$E_{6(-26)}$ 27	$E_{7(7)}$	$E_{6(6)}$
below not CLA				
$so^*(10)$	$u(5)$	$su(3,1) \oplus su(2)$ 13	$so(5,5)$	$sl(4,\mathbb{R}) \oplus sl(2,\mathbb{R})$
$E_{6(-14)}$	$so(10) \oplus so(2)$	$su(5,1)$ 21	$E_{6(6)}$ $E_{6(2)}$	$sl(6,\mathbb{R})$ $su(3,3)$
F'_4	$sp(3) \oplus su(2)$	$sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R})$ 20		
F''_4	$so(9)$	$so(7)$ 15		
$G_{2(2)}$	$su(2) \oplus su(2)$	0 min. $sl(2,\mathbb{R})$ max. 6 min. 5 max.		

where we display only the semisimple part $\mathcal{K}'$ of $\mathcal{K}$; $sl(n,\mathbb{C})_{\mathbb{R}}$ denotes $sl(n,\mathbb{C})$ as a real Lie algebra, (thus, $(sl(n,\mathbb{C})_{\mathbb{R}})^{\mathbb{C}} = sl(n,\mathbb{C}) \oplus sl(n,\mathbb{C})$); e_6 denotes the compact real form of E_6; and we have imposed restrictions to avoid coincidences or degeneracies due to well known isomorphisms: $so(1,2) \cong sp(1,\mathbb{R}) \cong su(1,1)$, $so(2,2) \cong so(1,2) \oplus so(1,2)$, $su(2,2) \cong so(4,2)$, $sp(2,\mathbb{R}) \cong so(3,2)$, $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an *Iwasawa decomposition* $G = KA_0N_0$, where A_0 is Abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a *minimal parabolic subgroup* of G. A *parabolic subgroup* $P = M'A'N'$ is any subgroup of G which contains a minimal parabolic subgroup.

Further, let $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$ denote the Lie algebras of G, K, P, M, A, N, resp.

For our purposes we need to restrict to *maximal parabolic subgroups* $P = MAN$, i.e.

rank$A = 1$, resp. to *maximal parabolic subalgebras* $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with dim $\mathcal{A} = 1$.

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, parameterized by a real number d, called the *conformal weight* or energy.

Further, let μ fix a discrete series representation D^μ of M on the Hilbert space V_μ, or the finite-dimensional (non-unitary) representation of M with the same Casimirs.

We call the induced representation $\chi = \text{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [DMPPT]. (These are called *generalized principal series representations* (or *limits thereof*) in [Knapp].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\,\mathcal{F}(g)\}$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the left regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g') , \quad g, g' \in G .$$

• An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}^{\mathbb{C}}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ.

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu v_0$.

For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

• Another main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets*. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines (arrows) between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation. The notion of multiplets was introduced and applied to representations of $SO_o(p,q)$ and $SU(2,2)$, resp., induced from their minimal parabolic subalgebras. Then it was applied to the conformal superalgebra, to infinite-dimensional (super-)algebras, to quantum groups.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the BGG Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m \ , \quad \beta^\vee \equiv 2\beta/(\beta, \beta)$$

ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. When the above holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [Dixmier]. More explicitly, [VKD1], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs). Then there exists [VKD1] an intertwining differential operator

$$\mathcal{D}_{m,\beta} \ : \ \mathcal{C}_{\chi(\Lambda)} \ \longrightarrow \ \mathcal{C}_{\chi(\Lambda - m\beta)}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \ = \ \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-})$$

where $\widehat{\mathcal{G}^-}$ denotes the right action on the functions $\mathcal{F}$.

In most of these situations the invariant operator $\mathcal{D}_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $\mathcal{G}$ is realized. Thus, studying the equations with trivial RHS:

$$\mathcal{D}_{m,\beta}\ f \ = \ 0 \ , \qquad f \in \mathcal{C}_{\chi(\Lambda)} \ ,$$

is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for $m = m_\beta = 1$, these equations are called conservation laws, and the elements $f \in \ker \mathcal{D}_{m,\beta}$ are called conserved currents.

The above construction works also for the subsingular vectors v_{ssv} of Verma modules. Such a vector is also expressed by a polynomial $\mathcal{P}_{ssv}(\mathcal{G}^-)$ in the universal enveloping algebra: $v^s_{ssv} = \mathcal{P}_{ssv}(\mathcal{G}^-)\, v_0$. Thus, there exists a *conditionally invariant differential operator* given explicitly by: $\mathcal{D}_{ssv} = \mathcal{P}_{ssv}(\widehat{\mathcal{G}^-})$, and a *conditionally invariant differential equation*. (Note that these operators (equations) are not of first order.)

Below in our exposition we shall use the so-called Dynkin labels:

$$m_i \ \equiv \ (\Lambda + \rho, \alpha_i^\vee) \ , \quad i = 1, \ldots, n,$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta) \ ,$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms. (Clearly, both the Dynkin labels and Harish-Chandra parameters have their origin in the BGG reducibility condition.)

Conformal algebras $so(n,2)$ and parabolically related

Let $\mathcal{G} = so(n,2)$, $n > 2$. We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\begin{aligned} \chi \ &= \ \{\, n_1\,, \ldots, n_{\tilde{h}}\,;\, c\,\}\ , \quad n_j \in \mathbb{Z}/2\ , \quad c = d - \tfrac{n}{2}\ , \\ &|n_1| < n_2 < \cdots < n_{\tilde{h}}\ , \quad n \text{ even}\ , \\ &0 < n_1 < n_2 < \cdots < n_{\tilde{h}}\ , \quad n \text{ odd}\ , \end{aligned}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first $\tilde{h}$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M} \cong so(n-1,1)$.

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simple intuitive way:

$$\begin{aligned}
\chi_1^{\pm} &= \{\epsilon n_1\,,\ldots,\, n_{\tilde h}\,;\, \pm n_{\tilde h+1}\}\ , \qquad n_{\tilde h} < n_{\tilde h+1}\ , \\
\chi_2^{\pm} &= \{\epsilon n_1\,,\ldots,\, n_{\tilde h-1}\,,\, n_{\tilde h+1}\,;\, \pm n_{\tilde h}\} \\
\chi_3^{\pm} &= \{\epsilon n_1,\ldots,n_{\tilde h-2},n_{\tilde h},n_{\tilde h+1}\,;\, \pm n_{\tilde h-1}\} \\
&\ldots \\
\chi_{\tilde h}^{\pm} &= \{\epsilon n_1\,, n_3\,,\ldots,\, n_{\tilde h}\,,\, n_{\tilde h+1}\,;\, \pm n_2\} \\
\chi_{\tilde h+1}^{\pm} &= \{\epsilon n_2\,,\ldots,\, n_{\tilde h}\,,\, n_{\tilde h+1}\,;\, \pm n_1\}
\end{aligned}$$

$$\epsilon = \begin{cases} \pm\,, & n\ even \\ 1, & n\ odd \end{cases}$$

Further, we denote by $\tilde{\mathcal{C}}_i^{\pm}$ the representation space with signature $\chi_i^{\pm}$.

The number of ERs in the corresponding multiplets is equal to:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|\,/\,|W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| \ = \ 2(1+\tilde h)$$

where $\mathcal{H}^{\mathbb{C}}$, $\mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$, $\mathcal{M}^{\mathbb{C}}$, resp. This formula is valid for the main multiplets of all conformal Lie algebras.

We show some examples of diagrams of invariant differential operators for the conformal groups $so(5,1)$, resp. $so(4,2)$, in 4-dimensional Euclidean, resp. Minkowski, space-time. Here and below we use the fact that algebras $so(p,q)$ for $p+q$-fixed are parabolically related. In Fig. 1. we show the simplest example for the most common using well known operators. In Fig. 2. we show the same example but using the group-theoretical parity splitting of the electromagnetic current, cf. [DoPe:78]. In Fig. 3. we show the general classification for $so(5,1)$ given in [DoPe:78]. These diagrams are valid also for $so(4,2)$ and for $so(3,3) \cong sl(4,\mathbb{R})$.

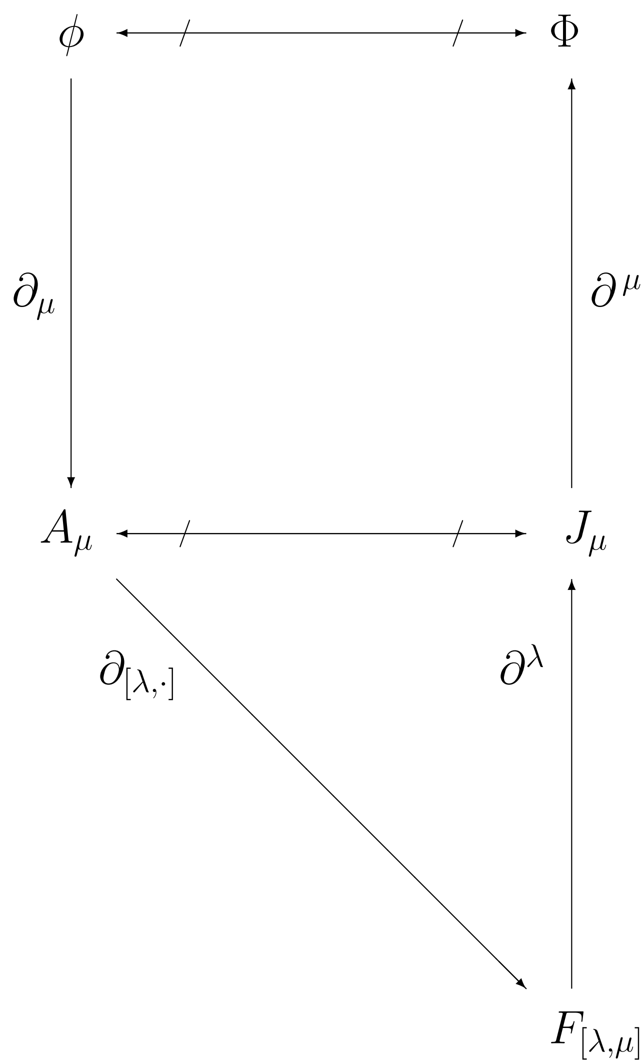


Fig. 1. Simplest example of diagram with conformal invariant operators (arrows are differential operators, dashed arrows are integral operators)

$$\partial_\mu = \frac{\partial}{\partial x_\mu}\,,\ \ A_\mu \text{ electromagnetic potential,}\quad \partial_\mu\,\phi = A_\mu$$

$$F \text{ electromagnetic field,}\quad \partial_{[\lambda}A_{\mu]} = \partial_\lambda A_\mu - \partial_\mu A_\lambda \ = \ F_{\lambda\mu}$$

$$J_\mu \text{ electromagnetic current,}\ \ \partial^\lambda F_{\lambda\mu} = J_\mu,\ \ \partial^\mu J_\mu = \Phi$$

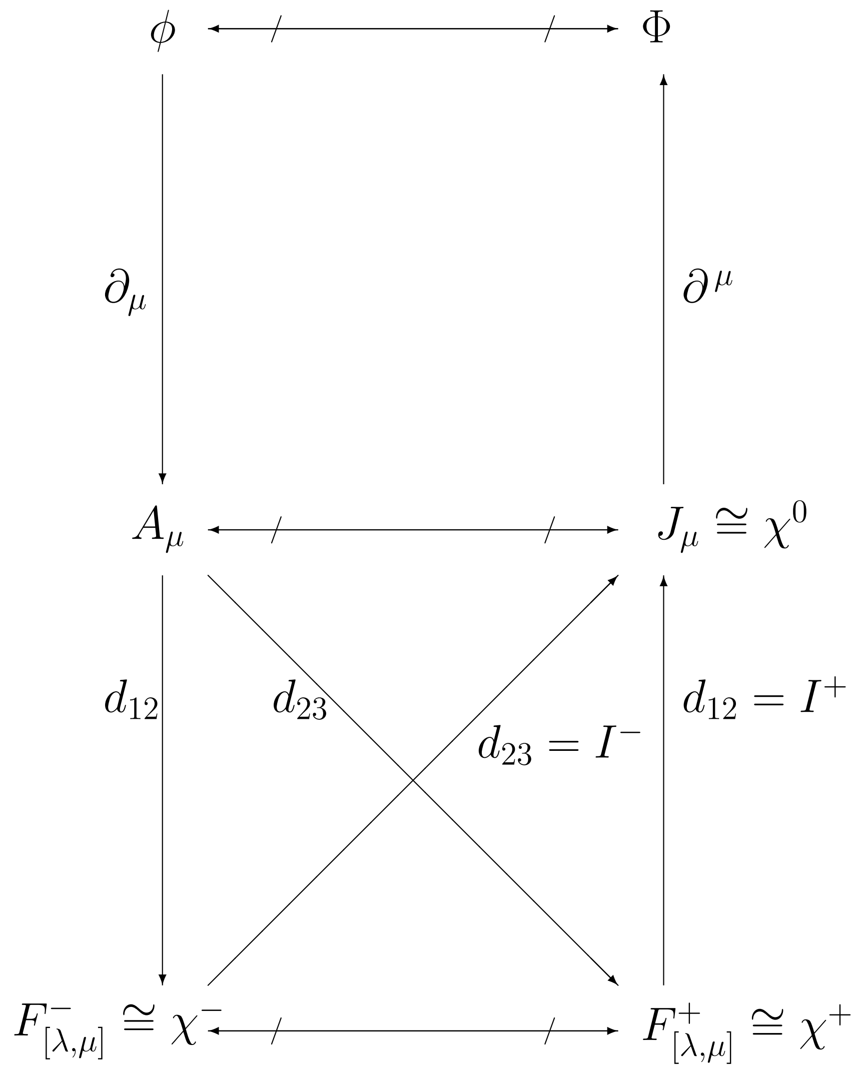


Fig. 2. More precise showing of the simplest example,
$F = F^+ \oplus F^-$ shows the parity splitting of the electromagnetic field,
d_{12}, d_{23} linear invariant operators corresponding to the roots α_{12}, α_{23}

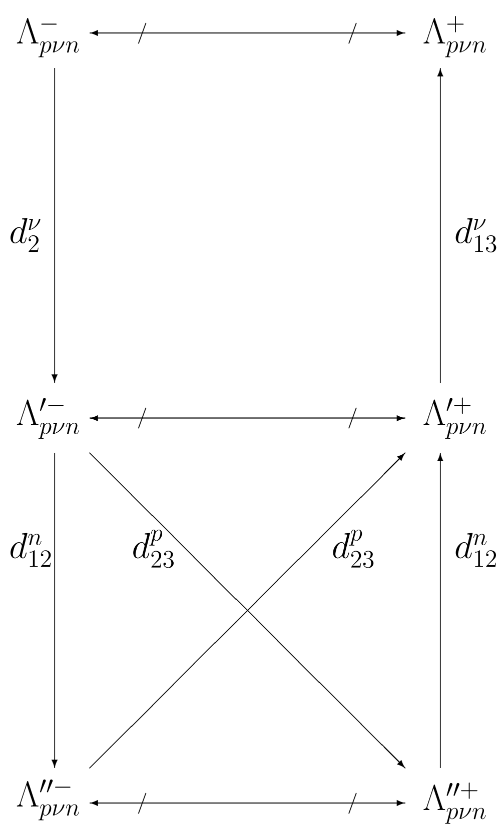


Fig. 3. The general classification of invariant differential operators valid for $so(4,2)$, $so(5,1)$ and $so(3,3) \cong sl(4,\mathbb{R})$.

p, ν, n are three natural numbers, the shown simplest case is when $p = \nu = n = 1$,

d_2^ν, d_{13}^ν linear invariant operators of order ν corresponding to the roots α_2, α_{13}

d_{12}^n, d_{23}^p linear invariant operators of order n, p corresponding to the roots α_{12}, α_{23}

Next in Fig. 4. we show the general even case $so(p,q)$, $p+q=2h+2$-even, while in Fig. 5. we show an alternative view of the same case:

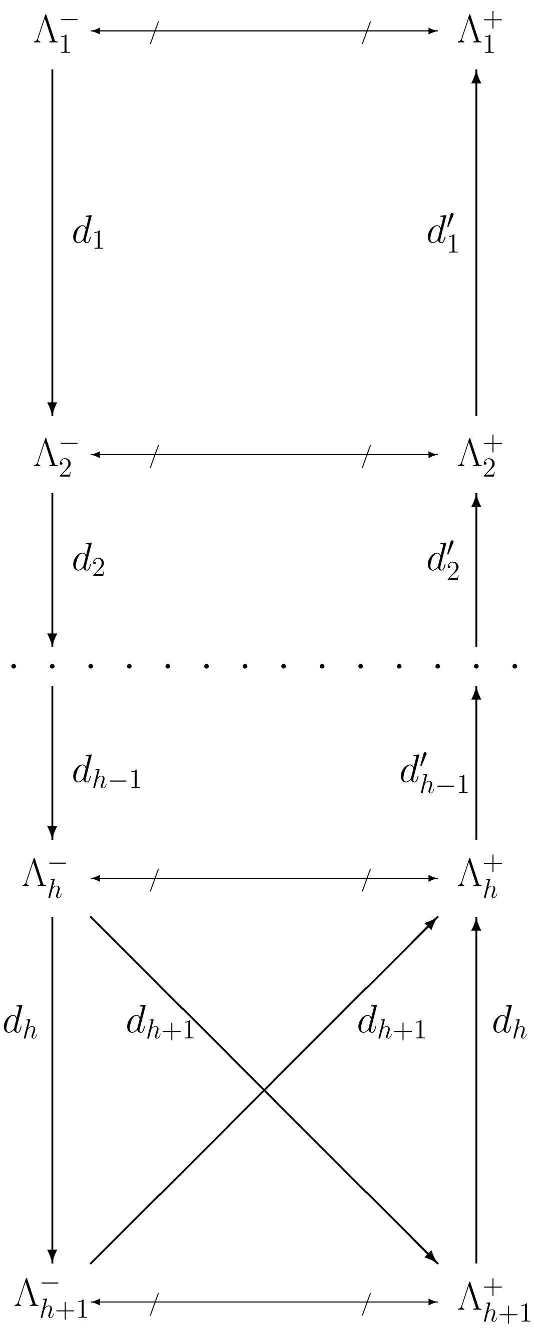


Fig. 4. The general classification of invariant differential operators in $2h$-dimensional space-time. By parabolic relation the diagram above is valid for all algebras $so(p,q)$, $p+q=2h+2$, even.

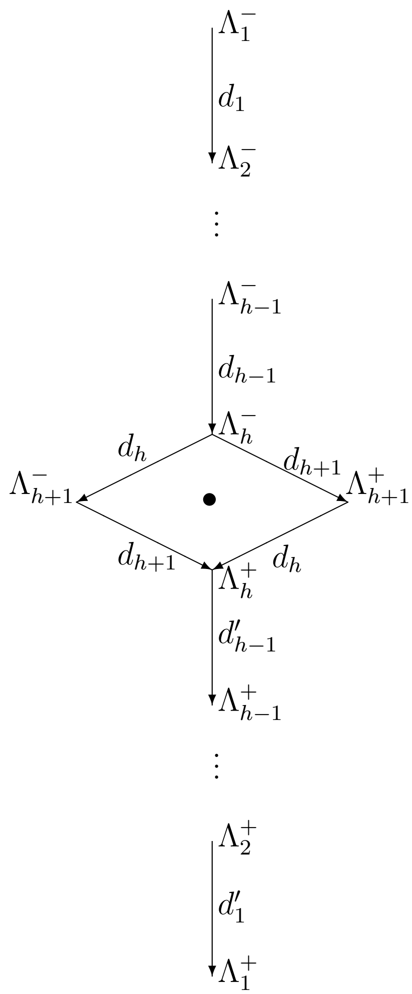


Fig. 5. Alternative showing of the case $so(p,q)$, $p+q=2h+2$, showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

Next in Fig. 6. we show the general odd case $so(p,q)$, $p+q = 2h+3$-odd, while in Fig. 7. we show an alternative view of the same case:

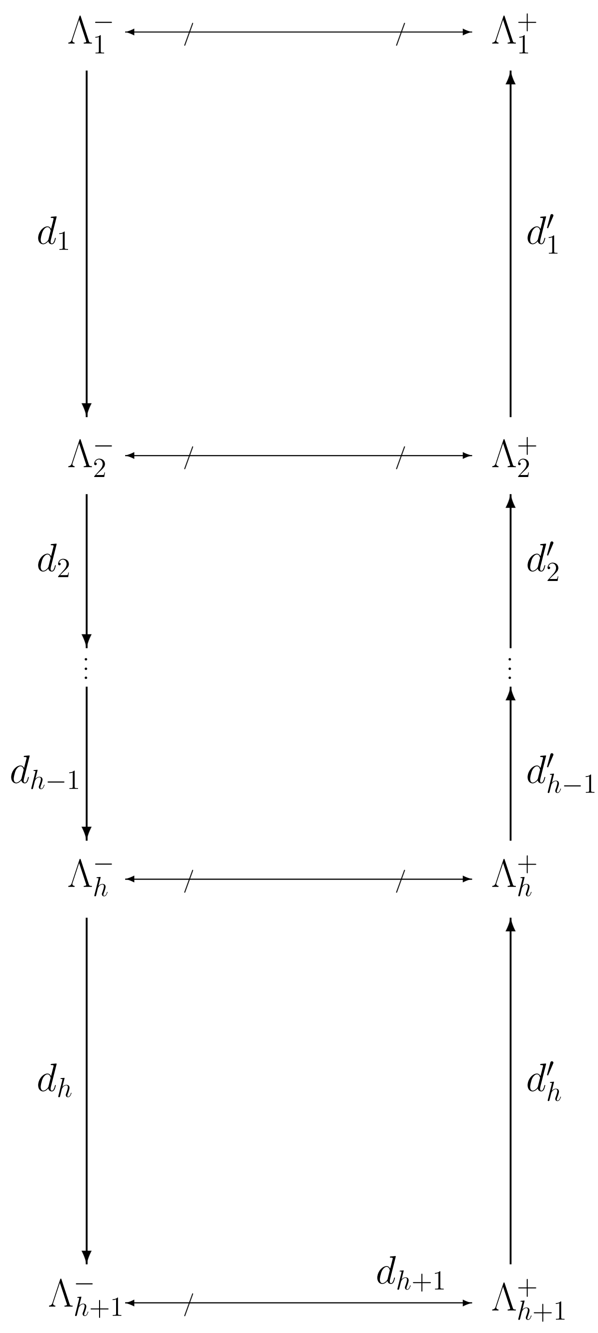


Fig. 6. The general classification of invariant differential operators in $2h+1$ dimensional space-time. By parabolic relation the diagram above is valid for all algebras $so(p,q)$, $p+q=2h+3$, odd.

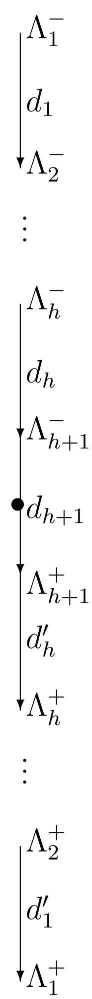


Fig. 7. Alternative showing of the case $so(p,q)$, $p+q=2h+3$, showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

The ERs in the multiplet are related by intertwining integral and differential operators. The integral operators were introduced by Knapp and Stein. They correspond to elements of the restricted Weyl group of $\mathcal{G}$. These operators intertwine the pairs $\tilde{\mathcal{C}}_i^{\pm}$

$$G_i^{\pm} \ : \ \tilde{\mathcal{C}}_i^{\mp} \longrightarrow \tilde{\mathcal{C}}_i^{\pm} \ , \quad i \ = \ 1,\ldots,1+\tilde{h}$$

The intertwining differential operators correspond to non-compact positive roots of the root system of $so(n+2,\mathbb{C})$, cf. [Dob88]. [In the current context, compact roots of $so(n+2,\mathbb{C})$ are those that are roots also of the subalgebra $so(n,\mathbb{C})$, the rest of the roots are non-compact.] The degrees of these intertwining differential operators are given just by the differences of the c entries [Dobsrni]:

$$\deg d_i = \deg d'_i = n_{\tilde{h}+2-i} - n_{\tilde{h}+1-i}\,, \quad i = 1,\ldots,\tilde{h}\,,$$
$$\deg d_{\tilde{h}+1} = n_2 + n_1\,, \quad n \text{ even}$$

where d'_h is omitted from the first line for $(p+q)$ even.

Remark: Note that for n-odd the integral operator G^+_{h+1} from Λ^-_{h+1} to Λ^+_{h+1} actually degenerates to the differential operator d_{h+1}. This results from the fact that integral kernel of G^+_{h+1} is a generalized function which is singular, and after regularization a la Gelfand it degenerates to a differential operator.$\diamondsuit$

Matters are arranged so that in every multiplet only the ER with signature χ^-_1 contains a finite-dimensional nonunitary subrepresentation in a subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional unitary irrep of $so(n+2)$ with signature $\{n_1\,, \ldots, n_{\tilde{h}}\,, n_{\tilde{h}+1}\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+_1, and is the image of the operator G^-_1.

Although the diagrams are valid for arbitrary $so(p,q)$ $(p+q \geq 5)$ the contents is very different. We comment only on the ER with signature χ_1^+. In all cases it contains an UIR of $so(p,q)$ realized on an invariant subspace $\mathcal{D}$ of the ER χ_1^+. That subspace is annihilated by the operator G_1^-, and is the image of the operator G_1^+. (Other ERs contain more UIRs.)

If $pq \in 2\mathbb{N}$ the mentioned UIR is a discrete series representation. (Other ERs contain more discrete series UIRs.)

And if $q = 2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, in which are realized a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Note that the $\deg d_i$, $\deg d'_i$, are Harish-Chandra parameters corresponding to the non-compact positive roots of $so(n+2,\mathbb{C})$. From these, only $\deg d_1$ corresponds to a simple root, i.e., is a Dynkin label.

Remark: The case of 3D Euclidean conformal symmetry $so(4,1)$ was treated in detail in Chapter 7 of [DMPPT] Springer book. The diagram with four ERs of Fig. 6 was given there. The case of 3D Minkowskian conformal symmetry $so(3,2)$ was treated in detail later. $\diamondsuit$

Above we considered $so(n,2)$ for $n > 2$. The case $n = 2$ is reduced to $n = 1$ since $so(2,2) \cong so(1,2) \oplus so(1,2)$. The case $so(1,2)$ is special and must be treated separately. But in fact, it is contained in what we presented already. In that case the multiplets contain only two ERs which may be depicted by the top pair $\chi_1^\pm$ in the pictures that we presented. And they have the properties that we described for $so(n,2)$ with $n > 2$. The case $so(1,2)$ was given already in 1946-7 independently by Gel'fand et al and by Bargmann.

Lie algebras $su(n,n)$ and parabolically related

Let $\mathcal{G} = su(n,n)$, $n \geq 2$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{C})_{\mathbb{R}}$. The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| \;=\; \binom{2n}{n}$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi \;=\; \{\, n_1\,, \ldots, n_{n-1}\,, n_{n+1} \ldots, n_{2n-1}\,;\, c\,\}\;, \quad n_j \in \mathbb{N}$$

The Knapp–Stein restricted Weyl reflection is given by:

$$\begin{aligned}
&G_{KS} \;:\; \mathcal{C}_\chi \;\longrightarrow\; \mathcal{C}_{\chi'}\;,\\
&\chi' = \{(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^*; -c\}\\
&(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^* \;\doteq\\
&(n_{n+1}, \ldots, n_{2n-1}, n_1, \ldots, n_{n-1})
\end{aligned}$$

Below in Fig. 8 and in Fig. 9 we give the diagrams for $su(n,n)$ for $n = 3,4$. (The case

$n = 2$ is already considered since $su(2,2) \cong so(4,2)$.) These are diagrams also for the parabolically related $sl(2n,\mathbb{R})$, and for $n = 2k$ these are diagrams also for the parabolically related $su^*(4k)$.

We use the following conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j\dots k}$ encoding the root $\beta_{j\dots k}$ and the number $m_{\beta_{j\dots k}}$ which is involved in the BGG criterion.

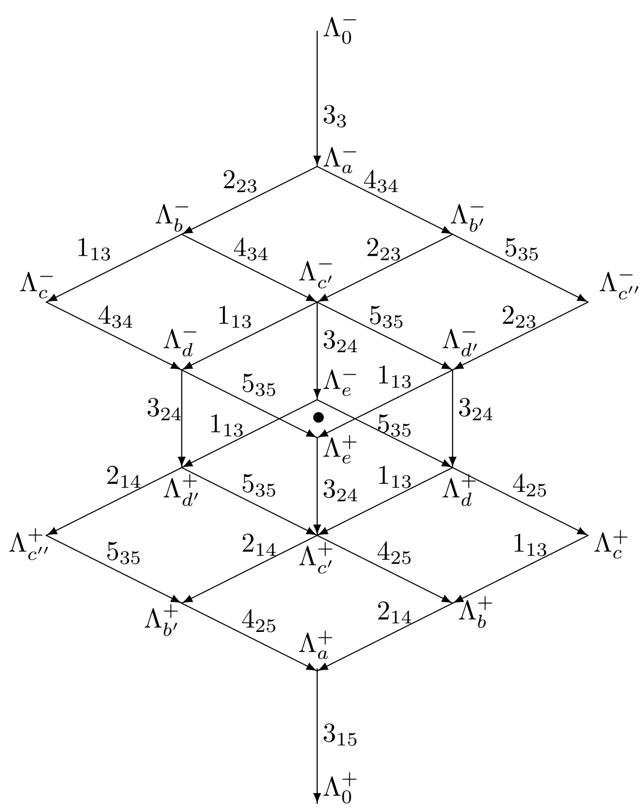


Fig. 8. Pseudo-unitary symmetry $su(3,3)$

The pseudo-unitary symmetry $su(n,n)$ is similar to conformal symmetry in n^2 dimensional space, for $n = 2$ coincides with conformal 4-dimensional case. By parabolic relation the $su(3,3)$ diagram above is valid also for $sl(6,R)$.

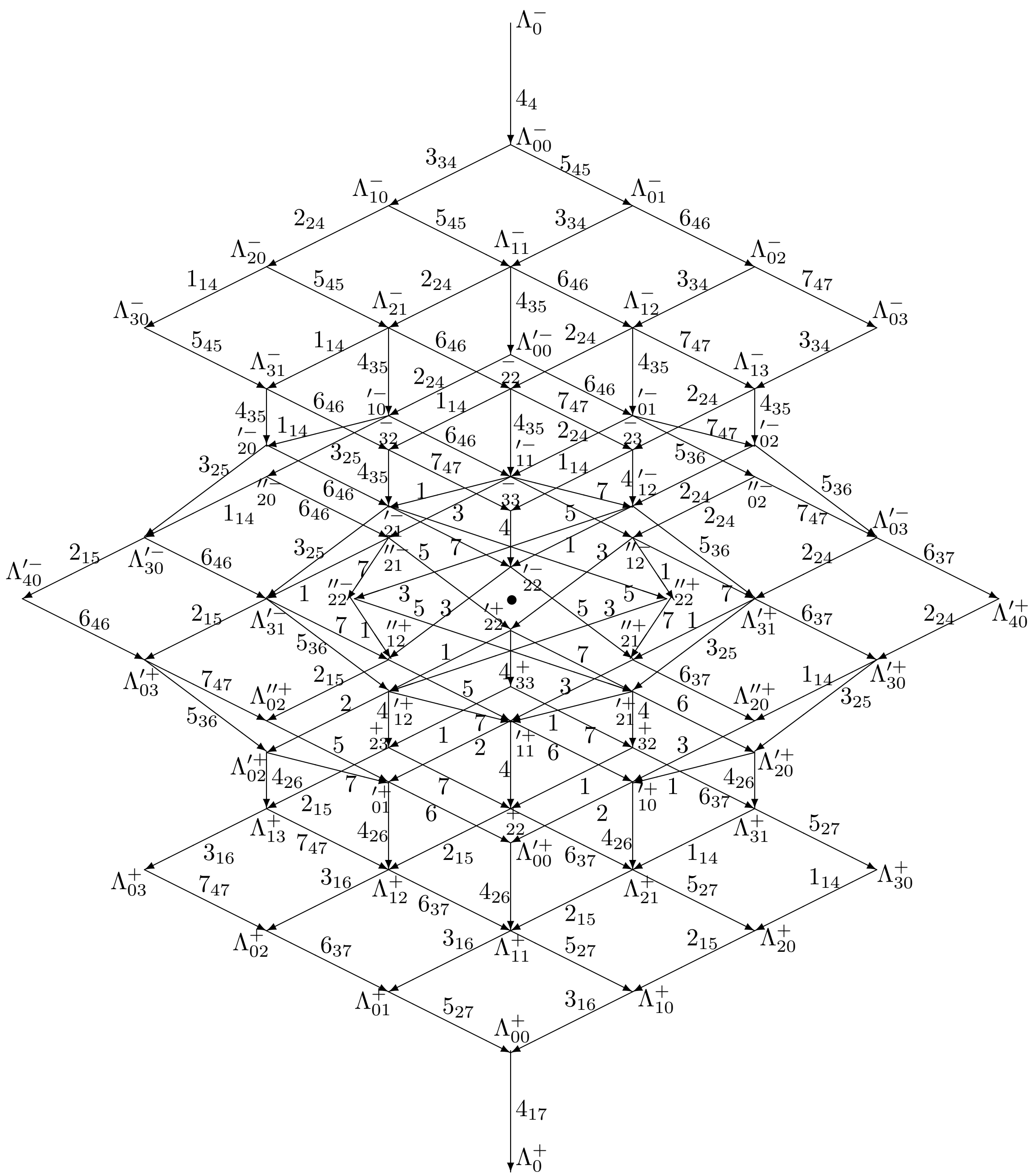


Fig. 9. Pseudo-unitary symmetry in 16-dimensional space.

By parabolic relation the $su(4,4)$ diagram above is valid also for $sl(8,R)$ and $su^*(8)$.

Lie algebras $sp(n,\mathbb{R})$ and $sp(\frac{m}{2},\frac{m}{2})$ (m–even)

Let $n \geq 2$. Let $\mathcal{G} = sp(n,\mathbb{R})$, the split real form of $sp(n,\mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{R})$. The number of ERs in the corresponding multiplets is:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| \;=\; 2^n$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi \;=\; \{\, n_1\,, \ldots, n_{n-1}\,;\, c\,\}\;, \qquad n_j \in \mathbb{N}\;,$$

The Knapp-Stein Weyl reflection acts as follows:

$$G_{KS} \;:\; \mathcal{C}_\chi \;\longrightarrow\; \mathcal{C}_{\chi'}\;, \chi' \;=\; \{\,(n_1,\ldots,n_{n-1})^*\,;\,-c\,\}\;,$$
$$(n_1,\ldots,n_{n-1})^* \;\doteq\; (n_{n-1},\ldots,n_1)$$

Below in Fig. 10, Fig. 11, Fig. 12 and Fig. 13 we give pictorially the multiplets for $sp(n,\mathbb{R})$ for $n = 3,4,5,6$. (The case $n = 2$ is already considered since $sp(2,\mathbb{R}) \cong so(3,2)$.) For $n = 2r$ these are also multiplets for $sp(r,r)$, $r = 1,2,3$. (The case $n = 2, r = 1$ is already considered due to $sp(1,1) \cong so(4,1)$ and the parabolic relation between $so(3,2)$ and $so(4,1)$.)

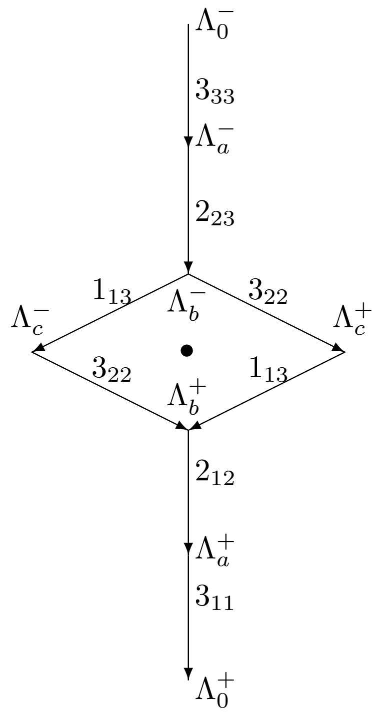


Fig. 10. Main multiplets for $Sp(3,I\!\!R)$

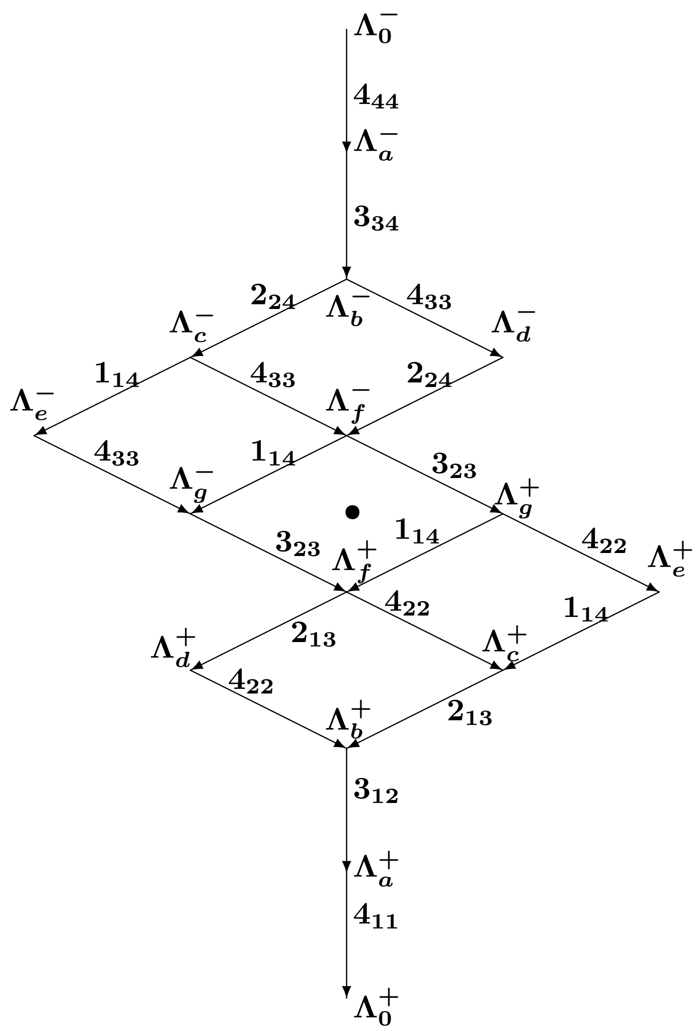


Fig. 11. Main multiplets for $sp(4, I\!R)$ and $sp(2, 2)$

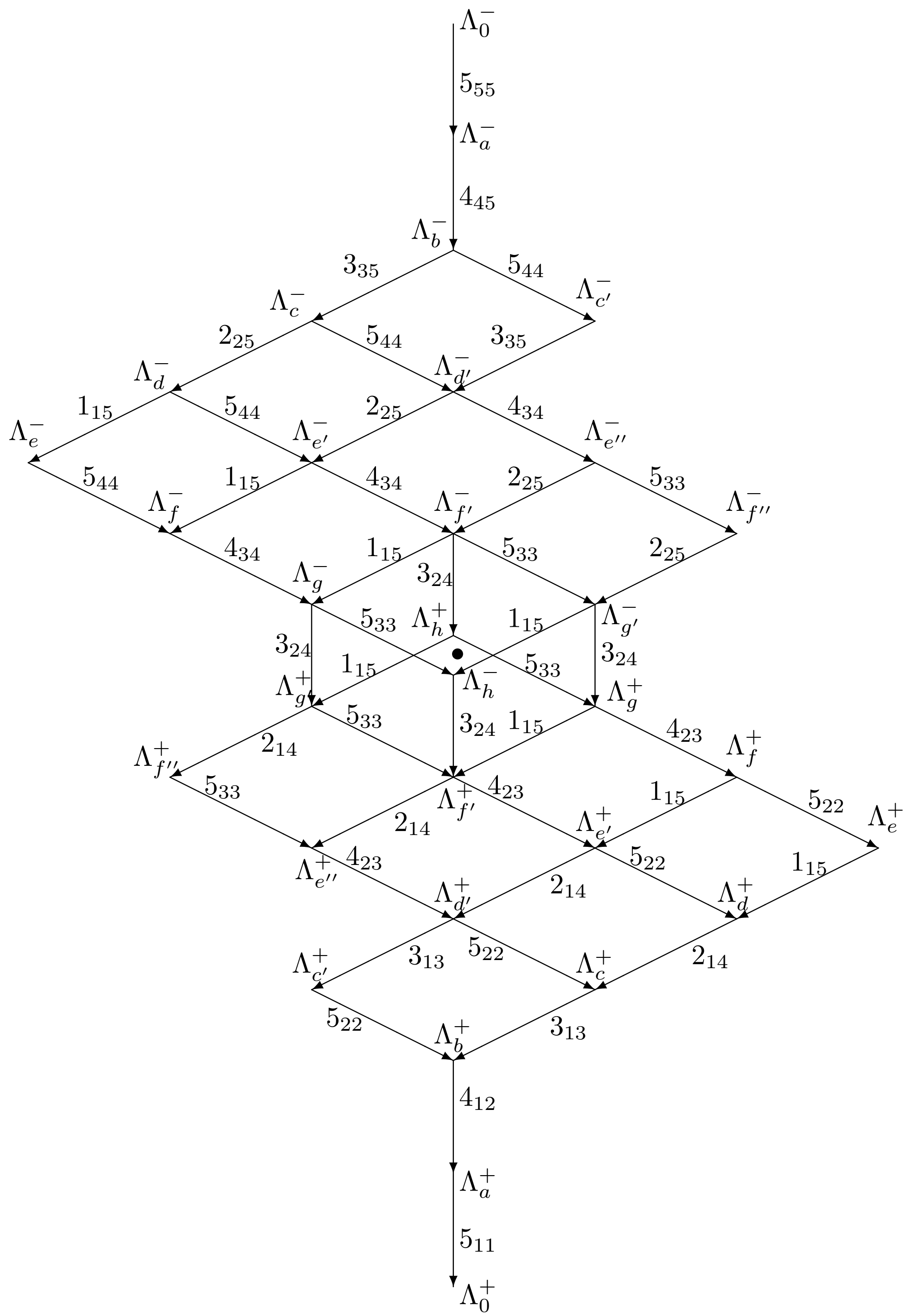


Fig. 12. Main multiplets for $Sp(5, I\!R)$

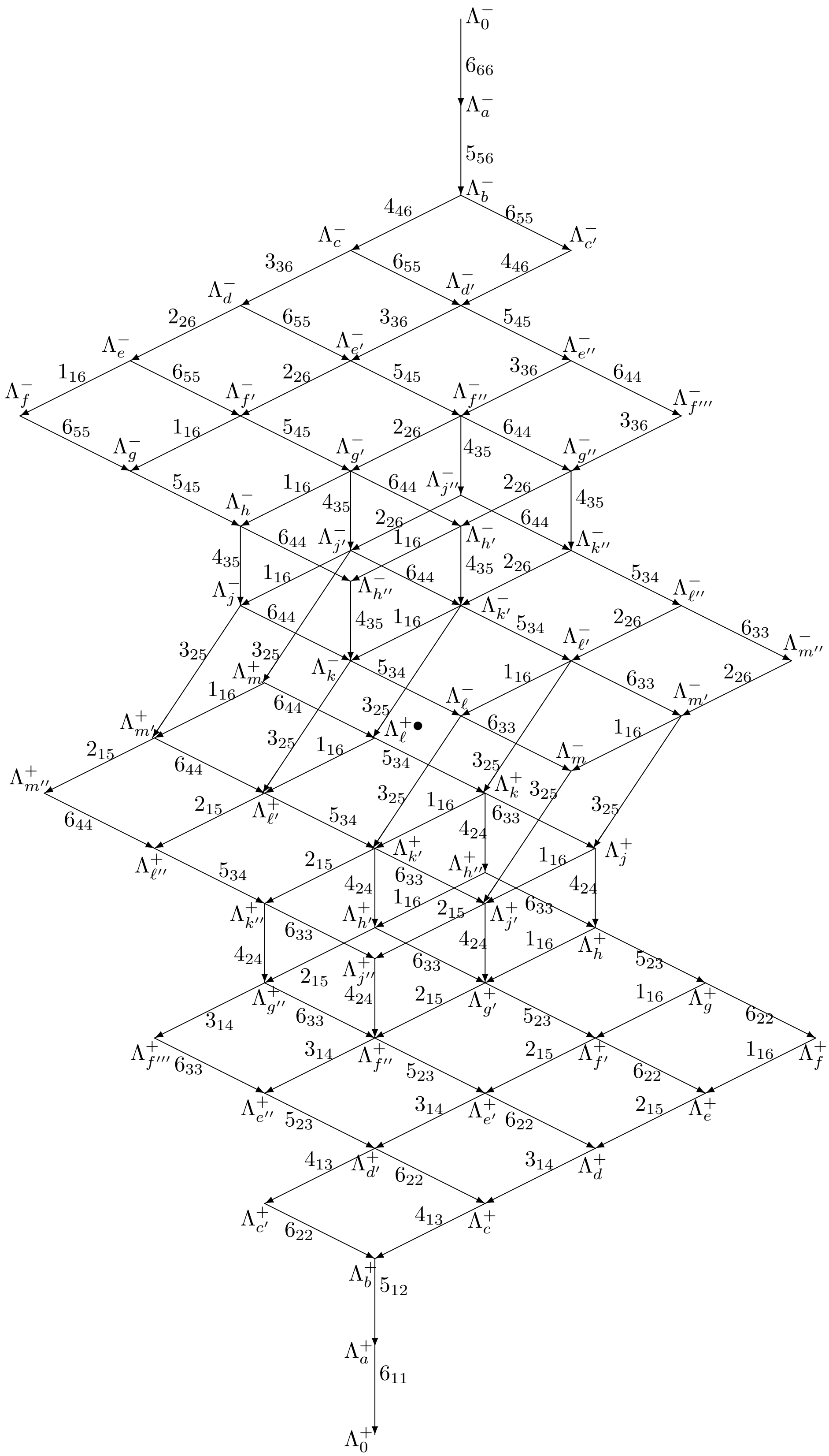
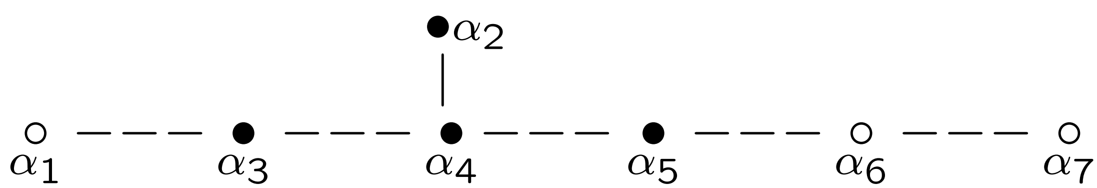


Fig. 13. Main multiplets for $sp(6, I\!R)$ and $sp(3,3)$.

Lie algebras $E_{7(-25)}$ and $E_{7(7)}$

Let $\mathcal{G} = E_{7(-25)}$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$, while $\mathcal{M} \cong E_{6(-6)}$. We choose a maximal parabolic subalgebra $\mathcal{P}_0 = \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0$, where $\mathcal{M}_0 = so(4)$, $\dim \mathcal{N}_0 = 51$.

The Satake diagram is:



The signatures of the ERs of $\mathcal{G}$ are:

$$\chi = \{ n_1 , \ldots , n_6 \, ; c \} , \qquad n_j \in \mathbb{N} .$$

expressed through the Dynkin labels:

$$n_i = m_i \ , \qquad c = -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_7) =$$
$$= -\tfrac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 +$$
$$+2m_6 + 2m_7)$$

The same signatures can be used for the parabolically related exceptional Lie algebra $E_{7(7)}$ (with $\mathcal{M}$-factor $E_{6(6)}$).

The noncompact roots of the complex algebra E_7 are:

$$\begin{aligned}
&\alpha_7\,,\quad \alpha_{17}\,,\quad \dots\,,\ \alpha_{67}\ ,\\
&\alpha_{1,37}\,,\ \alpha_{2,47}\,,\ \alpha_{17,4}\,,\ \alpha_{27,4}\,,\\
&\alpha_{17,34}\,,\ \alpha_{17,35}\,,\ \alpha_{17,36}\,,\ \alpha_{17,45}\,,\ \alpha_{17,46}\,,\\
&\alpha_{27,45}\,,\ \alpha_{27,46}\,,\\
&\alpha_{17,25,4}\,,\ \alpha_{17,26,4}\,,\ \alpha_{17,35,4}\,,\ \alpha_{17,36,4}\,,\\
&\alpha_{17,26,45}\,,\ \alpha_{17,36,45}\,,\\
&\alpha_{17,26,35,4}\,,\ \alpha_{17,26,45,4}\,,\\
&\alpha_{17,16,35,4}\ =\ \tilde{\alpha}\ ,
\end{aligned}$$

given through the simple roots α_i :

$$\begin{aligned}
&\alpha_{ij}\ =\ \alpha_i+\alpha_{i+1}+\cdots+\alpha_j\ ,\ i<j\ ,\\
&\alpha_{ij,k}=\alpha_{k,ij}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j+\alpha_k\ ,\qquad i<j\ ,
\end{aligned}$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of E_7, i.e., they will be labelled by the seven positive Dynkin labels $m_i \in \mathbb{N}$.

The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \ = \ 56$$

The multiplets are given in Fig. 14.

The Knapp-Stein operators $G^{\pm}_{\chi}$ act pictorially as reflection w.r.t. the bullet intertwining each $\mathcal{T}^{-}_{\chi}$ member with the corresponding $\mathcal{T}^{+}_{\chi}$ member.

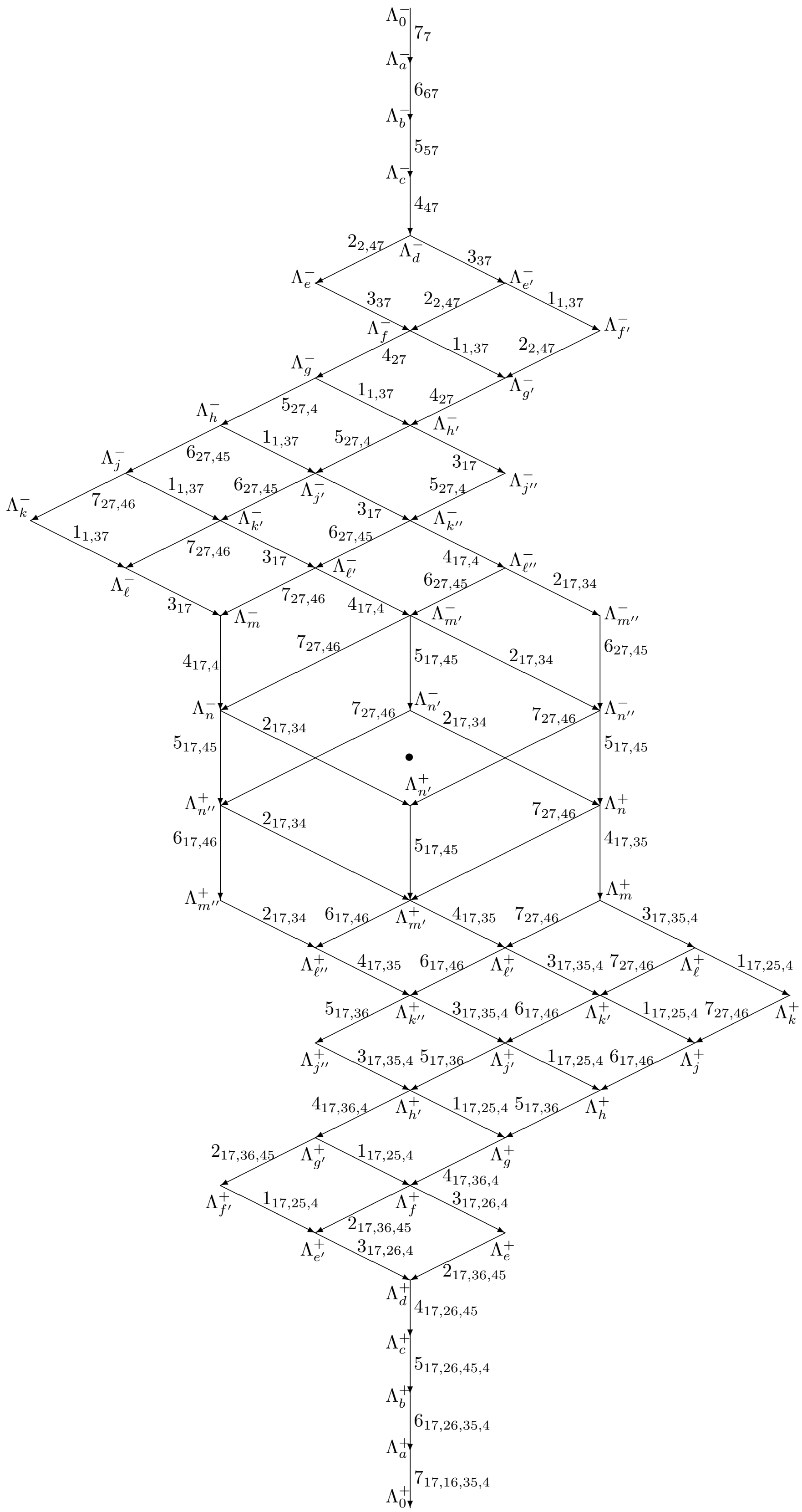
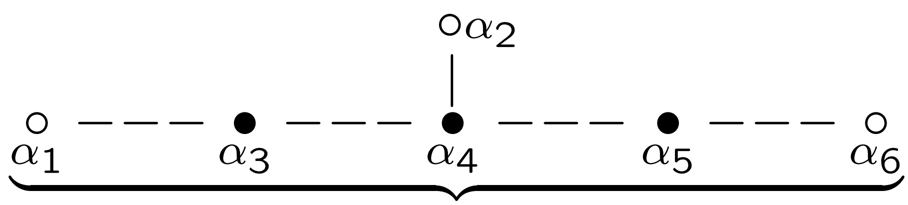


Fig. 14. Main multiplets for $E_{7(-25)}$ and $E_{7(7)}$.

Lie algebras $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$

Let $\mathcal{G} = E_{6(-14)}$. The maximal compact subalgebra is $\mathcal{K} \cong so(10) \oplus so(2)$, while $\mathcal{M} \cong su(5,1)$.

The Satake diagram is:



The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{\, n_1\,, n_3\,, n_4\,, n_5\,, n_6\,;\, c\} \ , \qquad c = d - \tfrac{11}{2} \ .$$

expressed through the Dynkin labels as:

$$n_i = m_i \ , \qquad -c \ = \ \tfrac{1}{2} m_{\tilde{\alpha}} \ = $$
$$= \ \tfrac{1}{2}(m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6)$$

The same signatures can be used for the parabolically related exceptional Lie algebras $E_{6(6)}$

and $E_{6(2)}$ with $\mathcal{M}$–factors $sl(6,\mathbb{R})$ and $su(3,3)$, resp.

Further, we need the noncompact roots of the complex algebra E_6 :

$$\begin{aligned}&\alpha_2\,,\ \alpha_{14}\,,\ \alpha_{15}\,,\ \alpha_{16}\,,\ \alpha_{24}\,,\ \alpha_{25}\,,\ \alpha_{26} \\ &\alpha_{2,4}\,,\ \alpha_{2,45}\,,\ \alpha_{2,46}\,,\ \alpha_{25,4}\,,\ \alpha_{15,4}\,,\ \alpha_{26,4} \\ &\alpha_{16,4}\,,\ \alpha_{15,34}\,,\ \alpha_{26,45}\,,\ \alpha_{16,34}\,,\ \alpha_{16,45} \\ &\alpha_{16,35}\,,\ \alpha_{16,35,4}\,,\ \alpha_{16,25,4}\ =\ \tilde{\alpha}\end{aligned} \tag{2}$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $\mathcal{G}$, i.e., they will be labelled by the six positive Dynkin labels $m_i \in \mathbb{N}$.

Since these algebras do not belong to the class of conformal Lie algebras (CLA) the number of ERs/GVMs in the multiplet is not given by the formula for the CLA cases. It turns out that each such multiplet contains 70 ERs/GVMs

- see Fig. 15. Another difference with the CLA class is that pictorially the the Knapp-Stein operators $G^{\pm}_{\chi}$ act as reflection w.r.t. the dotted line separating the $\mathcal{T}^{-}_{\chi}$ members from the $\mathcal{T}^{+}_{\chi}$ members (and not as reflection w.r.t. a central dot (bullet) as in the CLA cases).

Note that there are five cases when the embeddings correspond to the highest root $\tilde{\alpha}$: $V^{\Lambda^-} \longrightarrow V^{\Lambda^+}$, $\Lambda^+ = \Lambda^- - m_{\tilde{\alpha}}\,\tilde{\alpha}$. In these five cases the weights are denoted as: $\Lambda^{\pm}_{k''}$, $\Lambda^{\pm}_{k'}$, $\Lambda^{\pm}_{\tilde{k}}$, $\Lambda^{\pm}_{k}$, $\Lambda^{\pm}_{k^o}$, then: $m_{\tilde{\alpha}} = m_1, m_3, m_4, m_5, m_6$, resp. Thus, their action coincides with the action of the Knapp-Stein operators G^{+}_{χ} which in the above five cases degenerate to differential operators as we discussed for $so(q+1,q)$.

Note that the figure has the standard E_6 symmetry, namely, conjugation exchanging indices $1 \longleftrightarrow 6$, $3 \longleftrightarrow 5$.

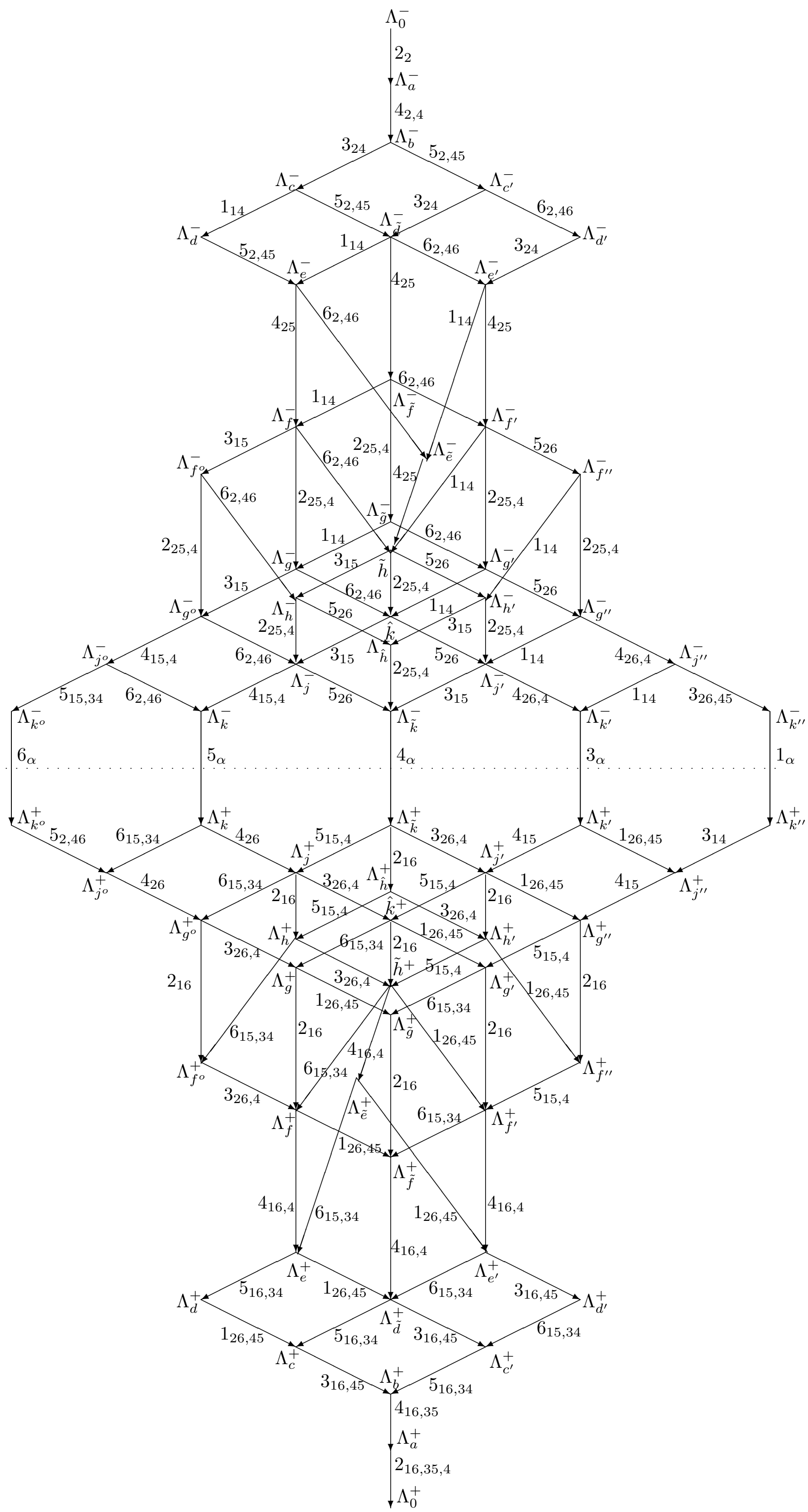


Fig. 15. Main multiplets for $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$.

Thank you for your attention!