Gravitational Waves in Non-Metric Gravity

Salvatore Capozziello

Università degli Studi di Napoli "Federico II"
and
Scuola Superiore Meridionale

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Open Questions

1. Do gravitational waves exist in a spacetime described by non-metric gravity?
2. What is their nature (scalar, vector or tensor) and polarization (transversal and/or longitudinal).
3. Are they massive or massless?
4. What is their velocity (c or less than c) ?
5. What helicity do they carry?
6. There are similarities between non-metric and teleparallel gravity?

Any good theory of gravity must predict the existence of gravitational waves.

In General Relativity, free structure-less particles move along timelike geodesic and their relative acceleration or tidal force is governed by geodesic deviation equation. Gravitational waves are massless, transverse and tensor waves of helicity two, travel at speed c and reproduce the two polarizations, plus and cross modes.

Curvature, Torsion and Non-Metricity I

The most general affine connection $\Gamma^{\alpha}{}_{\mu\nu}$ in metric-affine geometry, where curvature, torsion and non-metricity do not vanish, can be uniquely decomposed into three parts as

$$\boxed{\Gamma^{\alpha}{}_{\mu\nu} = \left\{ {}^{\alpha}_{\mu\nu} \right\} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}} \tag{1}$$

where $\left\{ {}^{\alpha}_{\beta\gamma} \right\}$ are the *Christoffel symbols* defined as

$$\boxed{\left\{ {}^{\rho}_{\mu\nu} \right\} \equiv \frac{1}{2} g^{\rho\alpha} \left(\partial_{\nu} g_{\mu\alpha} + \partial_{\mu} g_{\nu\alpha} - \partial_{\alpha} g_{\mu\nu} \right)} \tag{2}$$

$K^{\alpha}{}_{\mu\nu}$ is the *contorsion tensor* defined through the torsion tensor $T^{\alpha}{}_{\mu\nu}$

$$T^{\alpha}{}_{\mu\nu} = 2\Gamma^{\alpha}{}_{[\mu\nu]} \tag{3}$$

as

$$\boxed{K^{\alpha}{}_{\mu\nu} := \frac{1}{2} g^{\alpha\lambda} \left(T_{\mu\lambda\nu} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu} \right)} \tag{4}$$

Curvature, Torsion and Non-Metricity II

antisymmetric in the first and third indices

$$K_{\alpha\mu\nu} = -K_{\nu\mu\alpha} \tag{5}$$

whose antisymmetric part is given by

$$K^{\alpha}{}_{[\mu\nu]} = \frac{1}{2} T^{\alpha}{}_{\mu\nu} \tag{6}$$

while $L^{\alpha}{}_{\mu\nu}$ is the *disformation tensor*, symmetric in the second and third indices

$$\boxed{\begin{aligned} L^{\alpha}{}_{\mu\nu} &:= -\frac{1}{2} g^{\alpha\lambda} \left(Q_{\mu\lambda\nu} + Q_{\nu\lambda\mu} - Q_{\lambda\mu\nu} \right) \\ &= \frac{1}{2} Q^{\alpha}{}_{\mu\nu} - Q_{(\mu}{}^{\alpha}{}_{\nu)} \end{aligned}} \tag{7}$$

$$L^{\alpha}{}_{[\mu\nu]} = 0 \tag{8}$$

Curvature, Torsion and Non-Metricity III

here $Q_{\alpha\mu\nu}$ is the *non-metricity tensor* defined as

$$\boxed{Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\beta{}_{\alpha\mu} g_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu} g_{\beta\mu}} \tag{9}$$

symmetric in the last two indices

$$Q_{\alpha[\mu\nu]} = 0 \tag{10}$$

The *non-metricity scalar* is defined as

$$\boxed{\begin{aligned} Q &\equiv -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\gamma\beta\alpha} + \frac{1}{4} Q_\alpha Q^\alpha - \frac{1}{2} Q_\alpha \widetilde{Q}^\alpha \\ &= g^{\mu\nu}\left(L^\alpha{}_{\beta\mu} L^\beta{}_{\nu\alpha} - L^\alpha{}_{\beta\alpha} L^\beta{}_{\mu\nu}\right) \end{aligned}} \tag{11}$$

while the two traces of non-metricity tensor are

$$Q_\alpha \equiv Q_\alpha{}^\mu{}_\mu \tag{12}$$

Curvature, Torsion and Non-Metricity IV

and

$$\widetilde{Q}^{\alpha} \equiv Q_{\mu}{}^{\mu\alpha} \tag{13}$$

Therefore the contractions of the disformation tensor $L^{\lambda}{}_{\alpha\beta}$ become

$$L^{\lambda}{}_{\alpha\lambda} = -\frac{1}{2}Q_{\alpha} \quad , \quad L^{\alpha\lambda}{}_{\lambda} = \frac{1}{2}Q^{\alpha} - \widetilde{Q}^{\alpha} \tag{14}$$

The *non-metricity conjugate tensor* is defined as

$$\boxed{P^{\alpha}{}_{\mu\nu} = \frac{1}{2}\frac{\partial Q}{\partial Q_{\alpha}{}^{\mu\nu}}} \tag{15}$$

Symmetric in the last two indices

$$P^{\alpha}{}_{\mu\nu} = P^{\alpha}{}_{(\mu\nu)} \tag{16}$$

Curvature, Torsion and Non-Metricity V

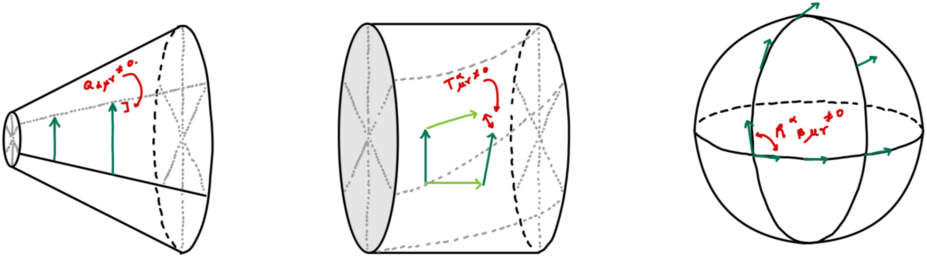
The superpotential $P^{\alpha}{}_{\mu\nu}$ defined in Eq. (15), thanks to the expression of the non-metricity scalar Q given in (11), can be written as

$$\begin{aligned} P^{\alpha}{}_{\mu\nu} &= \frac{1}{4}\left[-Q^{\alpha}{}_{\mu\nu} + 2Q_{(\mu}{}^{\alpha}{}_{\nu)} + Q^{\alpha} g_{\mu\nu} - \widetilde{Q}^{\alpha} g_{\mu\nu} - \delta^{\alpha}_{(\mu} Q_{\nu)}\right] \\ &= -\frac{1}{2} L^{\alpha}{}_{\mu\nu} + \frac{1}{4}\left(Q^{\alpha} - \widetilde{Q}^{\alpha}\right) g_{\mu\nu} - \frac{1}{4}\delta^{\alpha}_{(\mu} Q_{\nu)} \end{aligned} \tag{17}$$

This allows us to write the non-metricity scalar Q as

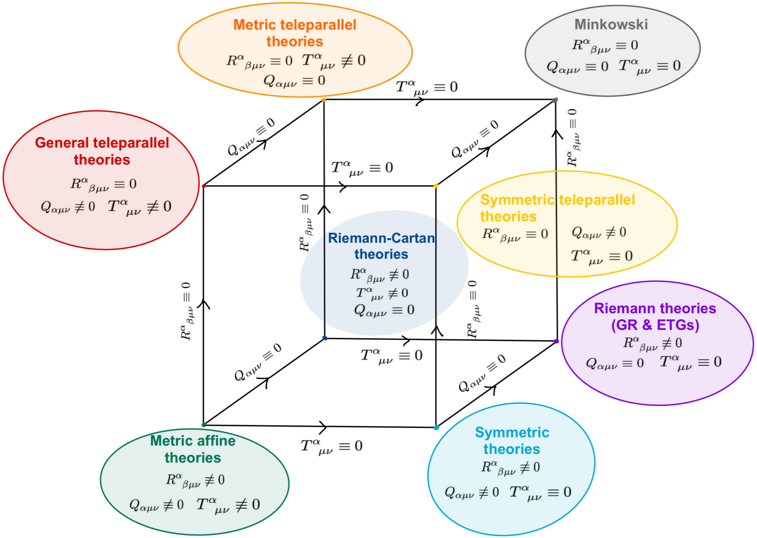
$$Q = Q_{\alpha\mu\nu} P^{\alpha\mu\nu} \tag{18}$$

Curvature, Torsion and Non-Metricity pictures

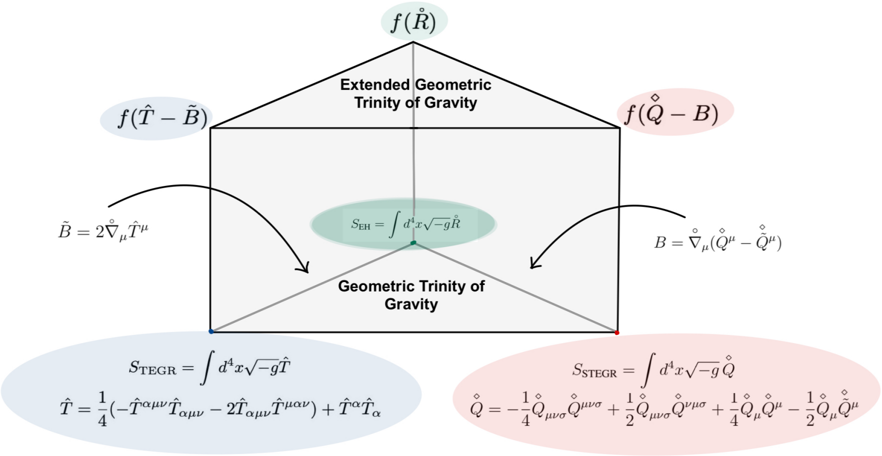


(A) (B) (C)

Metric-affine geometries



The Geometric Trinity of Gravity and its extensions



Consequences of non-metricity I

Consequences of non-metricity $Q_{\mu\alpha\beta} = \nabla_\mu g_{\alpha\beta} \neq 0$

1. It is not possible to raise and lower indices of an arbitrary four-vector A^α under the covariant derivative

$$g_{\alpha\beta}\nabla_\rho A^\beta \neq \nabla_\rho A_\alpha \quad \text{that is} \quad g_{\alpha\beta}\nabla_\rho A^\beta = \nabla_\rho A_\alpha - A^\beta Q_{\rho\alpha\beta} \tag{19}$$

therefore it is no longer true that

$$A_\alpha \nabla_\mu A^\alpha = A^\alpha \nabla_\mu A_\alpha \quad \text{but it is} \quad A_\alpha \nabla_\mu A^\alpha = A^\alpha \nabla_\mu A_\alpha - A^\alpha A^\beta Q_{\mu\alpha\beta} \tag{20}$$

2. The metric tensor $g_{\alpha\beta}$ is *not covariantly constant*, that is, in general, for two vector A^α and B^α parallel transported along a curve γ , their inner product does not remain constant under parallel transport along γ, whose tangent vector is $\boldsymbol{t}$, namely

$$\nabla_{\boldsymbol{t}}\left(A^\alpha B_\alpha\right) = \nabla_{\boldsymbol{t}}\left(g_{\alpha\beta}A^\alpha B^\beta\right) = A_\alpha \nabla_{\boldsymbol{t}} B^\alpha + B_\alpha \nabla_{\boldsymbol{t}} A^\alpha + A^\alpha B^\beta \nabla_{\boldsymbol{t}} g_{\alpha\beta} = A^\alpha B^\beta t^\sigma \nabla_\sigma g_{\alpha\beta} \neq 0 \tag{21}$$

Consequences of non-metricity II

Then the non-metricity, in general, does not preserve the length of vectors and the angles between two vectors parallel transported along the curve γ because

$$L(A) = |A| = \sqrt{A^\alpha A_\alpha} \rightarrow \nabla_t |A| \neq 0 \tag{22}$$

and

$$\cos\theta = \frac{A \cdot B}{|A||B|} \rightarrow \nabla_t \cos\theta \neq 0 \tag{23}$$

Symmetric Teleparallel Gravity

The *Symmetric Teleparallel Gravity* (STG) is a particular class of non-metric theories of gravity where curvature and torsion of the affine connection vanish, and therefore it is a formulation of gravity described only in terms of non-metricity, i.e.

$$\boxed{0 = R^{\lambda}{}_{\mu\nu\sigma}} = \Gamma^{\lambda}{}_{\mu\sigma,\nu} - \Gamma^{\lambda}{}_{\mu\nu,\sigma} + \Gamma^{\beta}{}_{\mu\sigma}\Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\beta}{}_{\mu\nu}\Gamma^{\lambda}{}_{\beta\sigma} \tag{24}$$

and

$$\boxed{T^{\alpha}{}_{\mu\nu} = 0} \tag{25}$$

The STG connection becomes

$$\boxed{\Gamma^{\alpha}{}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} + L^{\alpha}{}_{\mu\nu}} \tag{26}$$

Furthermore the non-metricity tensor $Q_{\alpha\mu\nu}$ satisfies the following Bianchi identity

$$\nabla_{[\alpha} Q_{\beta]\mu\nu} = 0 \tag{27}$$

The coincident gauge I

In STG theories, it is always possible to trivialize the connection, that is, we can choose the so-called *coincident gauge* where

$$\boxed{\Gamma^{\alpha}{}_{\mu\nu} = 0} \tag{28}$$

Indeed, the flatness of connection makes it integrable and therefore it can be written as

$$\Gamma^{\alpha}{}_{\mu\nu} = \left(\Lambda^{-1}\right)^{\alpha}{}_{\gamma}\,\partial_{\mu}\Lambda^{\gamma}{}_{\nu} \tag{29}$$

with the matrices $\Lambda^{\gamma}{}_{\nu} \in \mathrm{GL}(4,\mathbb{R})$, i.e., belonging to the general linear group. The absence of torsion of the connection gives

$$\partial_{[\beta}\Lambda^{\gamma}{}_{\nu]} = 0 \tag{30}$$

This implies that the transformation can be parameterized with a vector ξ^{μ} as

$$\Lambda^{\mu}{}_{\nu} = \partial_{\nu}\xi^{\mu} \tag{31}$$

Finally the connection is given by

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\beta}}\partial_{\mu}\partial_{\nu}\xi^{\beta} \tag{32}$$

The coincident gauge II

So it is always possible to choose a coordinate system, $\xi^\mu = x^\mu$, such that the connection vanishes. Physically, this means that the origin of tangent space coincides with the origin of spacetime. Hence, in concident gauge, we have

$$\boxed{\hat{\Gamma}^\alpha{}_{\mu\nu} = -L^\alpha{}_{\mu\nu}} \tag{33}$$

and the non-metricity tensor becomes

$$\boxed{Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}} \tag{34}$$

that is, the covariant derivative ∇, associated to the connection Γ, becomes the partial derivative ∂. It is worth noticing that $\nabla_\alpha g_{\mu\nu} \neq 0$, while only $\mathcal{D}_\alpha g_{\mu\nu} = 0$, where $\mathcal{D}_\alpha$, represents the covariant derivative associated to the Levi-Civita connection $\hat{\Gamma}^\alpha{}_{\mu\nu}$. We will use the coincident gauge when we will linearize the connection.

$f(Q)$ non-metric gravity

$f(Q)$ non-metric is an extension of STEGR expressed in the Palatini formalism, where $g_{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$ are independent dynamic variables, and f is an analytic function. The action is

$$S_{f(Q)} = \int_{\Omega} d^4x \left[\frac{1}{2\kappa^2}\sqrt{-g}f\left(Q\right) + \lambda_{\alpha}{}^{\beta\mu\nu}R^{\alpha}{}_{\beta\mu\nu} + \lambda_{\alpha}{}^{\mu\nu}T^{\alpha}{}_{\mu\nu} + \sqrt{-g}\mathcal{L}_m(g)\right] \tag{35}$$

where $\kappa^2 = 8\pi G/c^4$ and $\lambda_{\alpha}{}^{\beta\mu\nu} = \lambda_{\alpha}{}^{\beta[\mu\nu]}$, $\lambda_{\alpha}{}^{\mu\nu} = \lambda_{\alpha}{}^{[\mu\nu]}$ are Lagrange multipliers, i.e. other 96 plus 24 independent scalar fields. $R^{\alpha}{}_{\beta\mu\nu}$ and $T^{\alpha}{}_{\mu\nu}$ are the Riemann tensor and the torsion tensor, respectively. $\mathcal{L}_m$ is the material Lagrangian.

Field Equations in $f(Q)$ gravity I

Varying the action $S_{f(Q)}$ with respect to the metric tensor $g_{\mu\nu}$

$$\delta_g S_{f(Q)} = 0$$

and requiring that the variation of metric tensor vanishes on the boundary of domain Ω, we obtain the *field equations of $f(Q)$ gravity* in presence of matter. They are second-order nonlinear PDE

$$\boxed{\frac{2}{\sqrt{-g}}\nabla_\alpha\left(\sqrt{-g}f_Q P^\alpha{}_{\mu\nu}\right) - \frac{1}{2}g_{\mu\nu}f + f_Q\left(P_{\mu\alpha\beta}Q_\nu{}^{\alpha\beta} - 2Q^{\alpha\beta}{}_\mu P_{\alpha\beta\nu}\right) = \kappa^2 T_{\mu\nu}} \tag{36}$$

Varying the action (35) with respect to the STG connection $\Gamma^\alpha{}_{\mu\nu}$, the principle of last action

$$\delta_\Gamma S_{f(Q)} = 0$$

together with the vanishing of the variation of connection on the boundary, lead to the *connection equation of motion of $f(Q)$ gravity*

$$\boxed{\nabla_\mu\nabla_\nu\left(\sqrt{-g}f_Q P^{\mu\nu}{}_\alpha\right) = 0} \tag{37}$$

Field Equations in $f(Q)$ gravity II

The constraints of symmetric teleparallel theory are obtained by the vanishing of the following variations w.r.t. Lagrange multipliers, that is

$$\delta_{\lambda_\alpha{}^{\beta\mu\nu}} S_{f(Q)} = 0 \Rightarrow R^\alpha{}_{\beta\mu\nu} = 0 \tag{38}$$

while

$$\delta_{\lambda_\alpha{}^{\mu\nu}} S_{f(Q)} = 0 \Rightarrow T^\alpha{}_{\mu\nu} = 0 \tag{39}$$

Linearized equations in coincidence gauge I

Let us now linearly perturb Eq. (36) in absence of matter in the coincidence gauge. At first order in metric perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{40}$$

with $\eta_{\mu\nu}$, the Minkowski metric tensor, the *linearized field equations in vacuum* of Eq. (36), setting $T^{\mu}{}_{\nu} = 0$, reduce to

$$\boxed{f_Q(0)\Big[\Box h_{\mu\nu} - \left(\partial_\alpha\partial_\mu h^{\alpha}{}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}{}_{\mu}\right) - \eta_{\mu\nu}\left(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\right) + \partial_\mu\partial_\nu h\Big] = 0} \tag{41}$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the standard d'Alembert operator. Taking into account the following expansion of $f(Q)$ in terms of non-metricity scalar Q

$$f(Q) = f(0) + f_Q(0)Q + \mathcal{O}(Q^2) \tag{42}$$

Linearized equations in coincidence gauge II

the assumption $f(0) = 0$ implies in parameter expansion h

$$f(Q)^{(1)} = 0 \tag{43}$$
$$f(Q)^{(2)} = f_Q(0)Q^{(2)} \tag{44}$$

The *trace equation* of Eq. (41), coinciding with the linearized trace of Eq. (36), becomes

$$\boxed{\Box h = \partial_\alpha \partial_\beta h^{\alpha\beta}} \tag{45}$$

and Eq. (41) reduces to

$$\boxed{\Box h - 2\partial_{(\mu}\partial_\alpha h^\alpha{}_{\nu)} + \partial_\mu \partial_\nu h = 0} \tag{46}$$

a system of linear second order partial differential equations for perturbations $h_{\mu\nu}$. The linearized Eqs, (46) are *gauge invariant*. That is, if we perturb them to first order via an infinitesimal transformation $x'^\alpha = x^\alpha + \epsilon^\alpha(x)$, they remain unchanged.

Linearized equations in any gauge I

Let us now generalize our results without gauge fixing. Then, to linearize the non-metric gravity described by action (35), we perturb around the Minkowski spacetime both metric tensor and connection to first order, considered independent, as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad \Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha\,(0)}_{\mu\nu} + \Gamma^{\alpha\,(1)}_{\mu\nu} \tag{47}$$

where $|h_{\mu\nu}| \ll 1$ and $|\Gamma^{\alpha\,(1)}_{\mu\nu}| \ll 1$, with

$$\Gamma^{\alpha\,(0)}_{\mu\nu} = 0 \tag{48}$$

because the non-metric connection disappears at zero-order, when gravity is absent, to reproduce the flat spacetime. In any gauge, at first order in metric and connection perturbations, we get the following linear corrections for the non-metricity tensor $Q_{\alpha\mu\nu}$ and its contractions

$$Q^{(1)}_{\alpha\mu\nu} = \partial_\alpha h_{\mu\nu} - 2\,\Gamma^{(1)}_{(\mu|\alpha|\nu)} \tag{49}$$

$$Q^{\alpha\,(1)} = \partial^\alpha h - 2\,\Gamma^{\lambda}{}_{\lambda}{}^{\alpha\,(1)} \tag{50}$$

Linearized equations in any gauge II

$$\widetilde{Q}^{\alpha(1)} = \partial_\beta h^{\alpha\beta} - 2\,\Gamma^{(\beta}{}_{\beta}{}^{\alpha)(1)} \tag{51}$$

$$B^{\alpha(1)} = Q^{\alpha(1)} - \widetilde{Q}^{\alpha(1)} = \partial^\alpha h - \partial_\beta h^{\alpha\beta} + \Gamma^{\alpha}{}_{\beta}{}^{\beta(1)} - \Gamma^{\beta}{}_{\beta}{}^{\alpha(1)} \tag{52}$$

for the disformation tensor $L^\alpha{}_{\mu\nu}$

$$L^\alpha{}_{\mu\nu}{}^{(1)} = \frac{1}{2}\partial^\alpha h_{\mu\nu} - \partial_{(\mu} h^\alpha{}_{\nu)} + \Gamma^\alpha{}_{\mu\nu}{}^{(1)} \tag{53}$$

and for the non-metricity conjugate tensor $P^\alpha{}_{\mu\nu}$

$$\begin{aligned} P^\alpha{}_{\mu\nu}{}^{(1)} = &-\frac{1}{4}\partial^\alpha h_{\mu\nu} + \frac{1}{2}\partial_{(\mu} h^\alpha{}_{\nu)} + \frac{1}{4}\left(\partial^\alpha h - \partial_\beta h^{\alpha\beta}\right)\eta_{\mu\nu} - \frac{1}{4}\delta^\alpha_{(\mu}\partial_{\nu)} h \\ &+ \frac{1}{4}\left(\Gamma^\alpha{}_\beta{}^{\beta(1)} - \Gamma^\beta{}_\beta{}^{\alpha(1)}\right)\eta_{\mu\nu} - \frac{1}{2}\Gamma^\alpha{}_{\mu\nu}{}^{(1)} + \frac{1}{2}\delta^\alpha_{(\mu}\Gamma^\lambda{}_{\lambda|\nu)}{}^{(1)} \end{aligned} \tag{54}$$

Expanding $f(Q)$ as

$$f(Q) = f(0) + f_Q(0)Q + O(Q^2) \tag{55}$$

Linearized equations in any gauge III

we get

$$f(Q)^{(0)} = f(0) = 0 \quad \text{and} \quad f(Q)^{(1)} = f_Q(0) Q^{(1)} = 0 \tag{56}$$

$$f_Q^{(0)} = f_Q(0) \quad \text{and} \quad f_Q^{(1)} = 0 \tag{57}$$

assuming $f(0) = 0$, and $f(Q)^{(1)} = 0$ because in Q survive only second order terms in $h_{\mu\nu}$ and $\Gamma^{\alpha\,(1)}_{\mu\nu}$. Field Eqs. (36) in vacuum, at first order in h, become

$$\partial_\alpha P^{\alpha\mu}{}_\nu{}^{(1)} = 0 \tag{58}$$

while the connection Eqs. (37), always at the first order, yield

$$\partial_\mu \partial_\alpha P^{\alpha\mu}{}_\nu{}^{(1)} = 0 \tag{59}$$

which do not add any other constraints either on the metric or on the connection with respect to Eq. (58). Then the linearized field Eqs. (58) in vacuum in *any generic gauge* become

$$\Box h_{\mu\nu} - 2\,\partial^\alpha \partial_{(\mu} h_{\alpha|\nu)} + \partial_\mu \partial_\nu h + 2\,\partial_\alpha \Gamma^\alpha{}_{\mu\nu}{}^{(1)} - 2\,\partial_{(\mu} \Gamma^\lambda{}_{\lambda|\nu)}{}^{(1)} = 0 \tag{60}$$

Linearized equations in any gauge IV

However, taking into account the absence of torsion connection Γ, it becomes symmetric, namely

$$T^{\alpha}{}_{\mu\nu}[\Gamma] = 0 \Rightarrow \Gamma^{\alpha}{}_{[\mu\nu]} = 0 \tag{61}$$

result already used in the previous equations. Since we are in a symmetric teleparallel theory of gravity, the flatness of connection gives additional constraints on the connection, that is

$$R^{\alpha}{}_{\beta\mu\nu}[\Gamma] = 0 \Rightarrow \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu} = 0 \tag{62}$$

At first order, constraints (62) become

$$\Gamma^{\alpha}{}_{\beta\nu,\mu}{}^{(1)} = \Gamma^{\alpha}{}_{\beta\mu,\nu}{}^{(1)} \tag{63}$$

and contracting α and μ, we have

$$\partial_{\alpha}\Gamma^{\alpha}{}_{\beta\nu}{}^{(1)} = \partial_{\nu}\Gamma^{\alpha}{}_{\beta\alpha}{}^{(1)} \tag{64}$$

The symmetry of the connection implies

$$2\partial_{\alpha}\Gamma^{\alpha}{}_{\beta\nu}{}^{(1)} = 2\partial_{(\beta}\Gamma^{\alpha}{}_{\alpha|\nu)}{}^{(1)} \tag{65}$$

Linearized equations in any gauge V

Finally, from (65), Eq. (60) is further simplified taking the following form

$$\boxed{\Box h_{\mu\nu} - 2\,\partial^{\alpha}\partial_{(\mu}h_{\alpha|\nu)} + \partial_{\mu}\partial_{\nu}h = 0} \tag{66}$$

which are exactly the same differential equations obtained by linearizing the field equations of $f(Q)$ in coincident gauge in vacuum. **Since we get the same first order Eqs.** (66) **in** h**, we can avoid fixing the gauge**.

Wave solutions I

Wave solutions of Eqs. (66) can be obtained in Fourier formalism. Then, field Eqs (66) in Fourier space, according to the following waves expansion

$$h_{\mu\nu}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, \left(\widetilde{h}_{\mu\nu}(\vec{k}) e^{ik\cdot x} + c.c.\right) \tag{67}$$

becomes

$$\boxed{F_{\mu\nu} = k^2 \tilde{h}_{\mu\nu} - k_\mu k^\alpha \tilde{h}_{\alpha\nu} - k_\nu k^\alpha \tilde{h}_{\alpha\mu} + k_\mu k_\nu \tilde{h} = 0} \tag{68}$$

while its trace Eq. (45) in momentum space reads as

$$\boxed{k^2 \tilde{h} - k^\alpha k^\beta \tilde{h}_{\alpha\beta} = 0} \tag{69}$$

Now, let us suppose that the wave propagates in $+z$ direction with wave vector $k^\mu = \big(\omega, 0, 0, k_z\big)$, where $k^2 = \omega^2 - k_z^2$. Thus the wave expansion (67) reads as

$$h_{\mu\nu}(z,t) = \frac{1}{\sqrt{2\pi}} \int dk_z \Big(\tilde{h}_{\mu\nu}(k_z) e^{i(\omega t - k_z z)} + c.c.\Big) \tag{70}$$

Wave solutions II

where $c.c.$ stands for complex conjugate. The ten components of linear field Eqs. (68), in **k**-space, assume the form of a wave propagating along positive z-axis

$$
\begin{aligned}
F_{00} &= \left(\omega^2 + k_z^2\right)\tilde{h}_{00} + 2\omega k_z \tilde{h}_{03} - \omega^2 \tilde{h} = 0 \\
F_{01} &= k_z^2 \tilde{h}_{01} + \omega k_z \tilde{h}_{13} = 0 \\
F_{02} &= k_z^2 \tilde{h}_{02} + \omega k_z \tilde{h}_{23} = 0 \\
F_{03} &= \omega k_z \left(\tilde{h}_{00} - \tilde{h}_{33} - \tilde{h}\right) = 0 \\
F_{11} &= k^2 \tilde{h}_{11} = 0 \\
F_{12} &= k^2 \tilde{h}_{12} = 0 \\
F_{13} &= \omega k_z \tilde{h}_{01} + \omega^2 \tilde{h}_{13} = 0
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
F_{22} &= k^2 \tilde{h}_{22} = 0 \\
F_{23} &= \omega k_z \tilde{h}_{02} + \omega^2 \tilde{h}_{23} = 0 \\
F_{33} &= 2\omega k_z \tilde{h}_{03} + \left(\omega^2 + k_z^2\right)\tilde{h}_{33} + k_z^2 \tilde{h} = 0
\end{aligned}
\tag{72}
$$

Wave solutions III

while the linear trace Eq. (69) in the **k**-space becomes

$$\omega^2 \tilde{h}_{00} + 2\omega k_z \tilde{h}_{03} + k_z^2 \tilde{h}_{33} - \left(\omega^2 - k_z^2\right) \tilde{h} = 0 \tag{73}$$

where $\tilde{h}$ is the trace of metric perturbation $h_{\mu\nu}$ in the momentum space given by

$$\tilde{h} = \tilde{h}_{00} - \tilde{h}_{11} - \tilde{h}_{22} - \tilde{h}_{33} \tag{74}$$

The case $k^2 \neq 0$

We first solve the set of Eqs (71), (72) and Eq. (73) for $k^2 = M^2 \neq 0$. It is straightforward to obtain the following solution

$$
\begin{aligned}
\tilde{h}_{11} &= \tilde{h}_{12} = \tilde{h}_{22} = 0 \\
\tilde{h}_{13} &= -\frac{k_z}{\omega}\tilde{h}_{01} \\
\tilde{h}_{23} &= -\frac{k_z}{\omega}\tilde{h}_{02} \\
\tilde{h}_{33} &= -2\frac{k_z}{\omega}\tilde{h}_{03} - \frac{k_z^2}{\omega^2}\tilde{h}_{00}
\end{aligned}
\tag{75}
$$

with *four independent variables* $\tilde{h}_{01}$, $\tilde{h}_{02}$, $\tilde{h}_{03}$ and $\tilde{h}_{00}$. Therefore, when $k^2 \neq 0$, we obtain four degrees of freedom, which can be studied through the geodetic deviations.

The case $k^2 = 0$

Then in the case $k^2 = 0$ the solution of Eqs. (71),(72) and (73), where $\omega = k_z$, becomes

$$\begin{aligned} \tilde{h}_{22} &= -\tilde{h}_{11} \\ \tilde{h}_{13} &= -\tilde{h}_{01} \\ \tilde{h}_{23} &= -\tilde{h}_{02} \\ \tilde{h}_{33} &= -2\tilde{h}_{03} - \tilde{h}_{00} \end{aligned} \tag{76}$$

with *six independent variables* $\tilde{h}_{12}$, $\tilde{h}_{11}$, $\tilde{h}_{01}$, $\tilde{h}_{02}$, $\tilde{h}_{03}$ and $\tilde{h}_{00}$. Therefore six degrees of freedom. Even if $f(Q)$ gravity seems to have four or six degrees of freedom, it is possible to show, from the geodesic deviations, that only two propagate.

Polarization via geodesic deviation equation I

The displacement η^μ, which lies on the three-space orthogonal to the four-velocity u^α, can be chosen as $\eta^\mu = (0, \vec{\chi})$ where $\vec{\chi} = (\chi_x, \chi_y, \chi_z)$ is a spatial separation vector that connects two neighboring particles with non-relativistic velocity at rest in the freely falling local frame. We set up a quasi-Lorentz, normal coordinate system with origin on one particle and spatial coordinate χ^i for the other. The spatial components of the equation of geodesic deviation to the first order in the metric perturbation $h_{\mu\nu}$ are

$$\ddot{\chi}^i = -\mathcal{R}^{i(1)}_{\ 0j0}\chi^j \tag{77}$$

where i, j range over $(1, 2, 3)$, and $\mathcal{R}^{(1)}_{i0j0}$ are the electric components of the linearized Riemann tensor associated to Levi Civita connection. Equivalently, in STG, if we use the non-metricity related disformation tensor instead of the Riemann tensor, the geodesic deviation, in the local Lorentz frame, reduced to

$$\boxed{\ddot{\chi}^{\hat{i}} = 2\partial_{[\hat{j}} L^{\hat{i}}{}_{\hat{0}|\hat{0}]}{}^{(1)} \chi^{\hat{j}}}\,, \tag{78}$$

Polarization via geodesic deviation equation II

where the dot stands for the derivative with respect to coordinate time t and the hat over the indices stands for the components in the local inertial frame. In any gauge, from the linearized disformation tensor (53), we obtain

$$\partial_\nu {L^\mu}_{\alpha\beta}^{(1)} = \frac{1}{2}\left(\partial_\nu \partial^\mu h_{\alpha\beta} - 2\partial_\nu \partial_{(\alpha} {h^\mu}_{\beta)} + 2\partial_\nu {\Gamma^\mu}_{\alpha\beta}^{(1)}\right) , \tag{79}$$

and from Eq. (63), we find

$$2\partial_{[\nu} {L^\mu}_{\alpha|\beta]}^{(1)} = \frac{1}{2}\left(\partial_\alpha \partial_\beta {h^\mu}_\nu + \partial_\nu \partial^\mu h_{\alpha\beta} - \partial^\mu \partial_\beta h_{\alpha\nu} - \partial_\nu \partial_\alpha {h^\mu}_\beta\right) , \tag{80}$$

that is, the contribution of linearized connection disappears. For our components in any gauge, we get

$$2\partial_{[j} {L^i}_{0|0]}^{(1)} = \frac{1}{2}\left(\partial_0 \partial_0 {h^i}_j + \partial_j \partial^i h_{00} - \partial^i \partial_0 h_{0j} - \partial_j \partial_0 {h^i}_0\right) , \tag{81}$$

Polarization via geodesic deviation equation III

that is, exactly the opposite of the linearized Riemann tensor $\mathcal{R}^{i(1)}_{\ 0j0}$ expressed in LC connection. From the gauge invariance of Eq. (81), we have

$$2\partial_{[j}L^{i}{}_{0|0]}{}^{(1)} = 2\partial_{[\hat{j}}L^{\hat{i}}{}_{\hat{0}|\hat{0}]}{}^{(1)} \,, \tag{82}$$

which allows us to put the equation (78) into following form

$$\boxed{\ddot{\chi}^{\hat{i}} = 2\partial_{[j}L^{i}{}_{0|0]}{}^{(1)}\chi^{\hat{j}}} \,, \tag{83}$$

which is the geodesic deviation equation in $f(Q)$ gravity, in the proper reference frame of freely falling particles where $\partial_{[j}L^{i}{}_{0|0]}{}^{(1)}$ is expressed in *any gauge*, as well as the metric perturbations $h_{\mu\nu}$ in Eq. (81). Eq. (83) can be regarded as the relative acceleration between two freely falling point particles.

Polarization via geodesic deviation equation IV

The linear system of differential Eqs. (83), for a wave traveling along positive z-axis in a local proper reference, reads as

$$\left\{\begin{array}{l} \ddot{\chi}_x = -\frac{1}{2}h_{11,00}\chi_x - \frac{1}{2}h_{12,00}\chi_y + \frac{1}{2}\left(h_{01,03} - h_{13,00}\right)\chi_z \\ \ddot{\chi}_y = -\frac{1}{2}h_{12,00}\chi_x - \frac{1}{2}h_{22,00}\chi_y + \frac{1}{2}\left(h_{02,03} - h_{23,00}\right)\chi_z \\ \ddot{\chi}_z = \frac{1}{2}\left(h_{01,03} - h_{13,00}\right)\chi_x + \frac{1}{2}\left(h_{02,03} - h_{23,00}\right)\chi_y + \frac{1}{2}\left(2h_{03,03} - h_{33,00} - h_{00,33}\right)\chi_z \end{array}\right. \tag{84}$$

In the case $k^2 = M^2 \neq 0$, imposing the following initial conditions, the initial displacement $\chi(0) = \mathbf{R} = (\chi_x^0, \chi_y^0, \chi_z^0)$ and the initial relative velocity $\dot{\chi}(0) = 0$, after double integration with respect to t of the system (84), we obtain the solution

$$\boxed{\chi_x(t) = \chi_x^0\ , \quad \chi_y(t) = \chi_y^0\ , \quad \chi_z(t) = \chi_z^0} \tag{85}$$

which is not a wave, that is, *there is no mode* associated with $k^2 \neq 0$. That is, when $k^2 \neq 0$, **none of the four degrees of freedom has a physical meaning**.

Polarization via geodesic deviation equation V

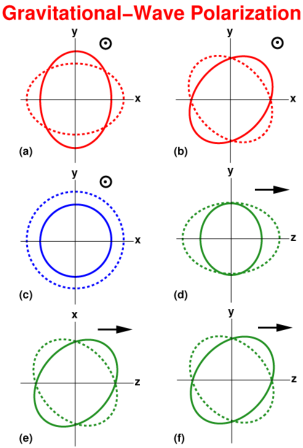
In the case $k^2 = 0$, that implies $\omega = k_z$, by double integration with respect to t of the equation of geodesic deviation to first order in the perturbation *at fixed* k_z, for a single plane wave, the system (84) yields as solution

$$\boxed{\left\{\begin{array}{l} \chi_x(t) = \chi_x^0 - \frac{1}{2}\left(\tilde{h}^{(+)}\chi_x^0 + \tilde{h}^{(\times)}\chi_y^0\right)e^{i\omega(t-z)} \\ \chi_y(t) = \chi_y^0 - \frac{1}{2}\left(\tilde{h}^{(\times)}\chi_x^0 - \tilde{h}^{(+)}\chi_y^0\right)e^{i\omega(t-z)} \\ \chi_z(t) = \chi_z^0 \end{array}\right.} \tag{86}$$

where $\tilde{h}_{11} = \tilde{h}^{(+)}$ and $\tilde{h}_{12} = \tilde{h}^{(\times)}$. **Hence, when $k^2 = 0$, only two degrees of freedom of the initial six survive**. This solution describes the response of a ring of masses hit by a gravitational wave. When we have the case $\tilde{h}^{(\times)} = 0$, from the solution (86), the effect of gravitational wave is to distort the circle of particles into ellipses oscillating in a $+$ pattern, while, in the case $\tilde{h}^{(+)} = 0$, the ring distorts into ellipses oscillating in a $\times$ pattern rotated by 45 degrees in a right handed sense with respect to it.

Polarization via geodesic deviation equation VI

In summary, we obtain the well-know plus and cross modes, massless, transverse of spin 2, typical of General Relativity, as seen in case (a) and (b) of the following figure,



Polarization via geodesic deviation equation VII

and the gravitational wave can be put into the form

$$h_{\mu\nu}(z,t) = \frac{1}{\sqrt{2\pi}} \int dk_z \left[\epsilon^{(+)}_{\mu\nu}\tilde{h}^{(+)}(\omega) + \epsilon^{(\times)}_{\mu\nu}\tilde{h}^{(\times)}(\omega)\right] e^{i\omega(t-z)} + c.c. \tag{87}$$

where $\epsilon^{(+)}_{\mu\nu}$ and $\epsilon^{(\times)}_{\mu\nu}$ are the polarization tensors defined as

$$\epsilon^{(+)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon^{(\times)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{88}$$

Hence, in $f(Q)$ gravity, in any gauge, we obtain the same gravitational waves predicted by General Relativity in the TT gauge. Finally a gravitational wave propagating along an arbitrary $\boldsymbol{k}$ direction in $f(Q)$ gravity, from Eq. (87), becomes

$$\boxed{h_{\mu\nu}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[\epsilon^{(+)}_{\mu\nu}\tilde{h}^{(+)}(\vec{k}) + \epsilon^{(\times)}_{\mu\nu}\tilde{h}^{(\times)}(\vec{k})\right] e^{i(\omega t - \vec{k}\cdot\vec{x})} + c.c.} \tag{89}$$

$f(Q)$ and $f(T)$ theories of gravity compared via gravitational waves I

Let us now take into account the action of $f(T)$ gravity in presence of standard matter

$$S_{f(T)} = \int_{\Omega} d^4x \left[\frac{1}{2\kappa^2} f(T) + \mathcal{L}_m\right] e \ . \tag{90}$$

where f is an analytic function anche T, the scalar torsion. The variation of the action (90) with respect to the vierbein fields $e^a{}_\rho$ yields the following field equations

$$\frac{4}{e}\partial_\sigma \left(e f_T S_a{}^{\rho\sigma}\right) + f(T)\, e_a{}^\rho - 4 f_T T^\mu{}_{\nu a} S_\mu{}^{\nu\rho} = 2\kappa^2 \mathcal{T}_a{}^\rho \ , \tag{91}$$

where $\mathcal{T}_a{}^\rho$ is the energy momentum tensor of matter defined as

$$\mathcal{T}_a{}^\rho = -\frac{1}{e}\frac{\delta\left(e\mathcal{L}_m\right)}{\delta e^a{}_\rho} \ . \tag{92}$$

and the superpotential $S^{\rho\mu\nu}$ defined as

$$S^{\rho\mu\nu} = \frac{1}{2}\left(K^{\mu\nu\rho} - g^{\rho\nu} T^{\sigma\mu}_{\sigma} + g^{\rho\mu} T^{\sigma\nu}_{\sigma}\right) \ , \tag{93}$$

$f(Q)$ and $f(T)$ theories of gravity compared via gravitational waves II

by means of the torsion tensor $T^{\nu}{}_{\rho\mu}$ and the contortion tensor $K^{\nu}{}_{\rho\mu}$. In the weak field limit, we expand the tetrad field around the flat geometry described by the trivial tetrad $e^{a}{}_{\mu} = \delta^{a}{}_{\mu}$ as follows

$$e^{a}{}_{\mu} = \delta^{a}{}_{\mu} + E^{a}{}_{\mu} \; , \tag{94}$$

where $|E^{a}{}_{\mu}| \ll 1$. Then, the linear perturbation of field Eqs. (91) becomes

$$\boxed{f_T^{(0)} \Box \bar{E}^{\rho}{}_{\tau} = -2\kappa^2 \mathcal{T}_{\tau}{}^{\rho(0)}} \; , \tag{95}$$

where

$$\bar{E}_{\mu\nu} = E_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} E \; . \tag{96}$$

In vacuum, Eq. (95) gives

$$\Box \bar{E}^{\rho}{}_{\tau} = 0 \tag{97}$$

that is, exactly the same waves predicted by General Relativity. Therefore, both in $f(Q)$ and in $f(T)$, gravitational waves exist, they are massless-transverse tensor and the polarization modes reproduce the plus and cross polarizations. **From gravitational waves, $f(Q)$ and $f(T)$ gravities are indistinguishable from General Relativity.**

Conclusions and Perspectives

1. Gravitational waves in $f(Q)$ gravity are the same as in General Relativity, i.e., massless, transverse, tensors with helicity two with plus and cross modes. The freely falling point-like particles follow timelike geodesic as in General Relativity, and the evolution of their separation vector is governed by the geodesic deviation equation of General Relativity.
2. From gravitational waves, $f(Q)$ and $f(T)$ gravity are indistinguishable from General Relativity.
3. Equivalence Principle can be recovered by coincident gauge. In general STGs, this is matter of debate.
4. The introduction of boundary terms allow to compare teleparallel and symmetric teleparallel theories with $f(R)$ gravity.
5. Using general connection Γ could give further gravitational wave modes.

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