Double space T-dualization of type II superstring with coordinate dependent RR field*

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Abstract

In this article we show the equivalence of the Buscher analytical T-duality approach and algebraic double space approach in one specific nontrivial case. The type II superstring with the coordinate dependent RR field strength is considered here. All other background fields are constant. Also we give arguments for the specific choice of the background fields and explain motivation for it. In addition the non(anti)commutativity relations are given. It is confirmed that such choice of the background does not produce the nonzero Poisson bracket of the fermionic coordinates.

1. Introduction

All five consistent superstring theories are connected by web of T- and S-dualities [1, 2, 3, 4]. Consequently, this fact suggests existence of one unique theory, in literature known as M-theory, and further, the deeper insight into string duality transformations. In this article we will explore some properties of T-duality.

T-duality is a phenomenon experienced just by extended objects, strings [3, 4, 5, 6, 7, 8]. Analytically T-duality is realized within Buscher procedure [5, 9]. The first step in obtaining the T-dual theory is localization of the global isometry (at least along one direction) introducing the corresponding world-sheet covariant derivatives. In this way gauge fields $v^\mu_\pm$ come into the story. But, initial and its T-dual theory must have the same number of the degrees of freedom, so we need to eliminate all newly introduced degrees of freedom using Lagrange multipliers y_μ. On the equations of motion for the gauge fields we obtain T-dual theory. One aggravating circumstance is that Buscher procedure breaks down when we have coordinate dependent

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background fields. But it is possible, at least in the cases where background infinitesimally depend on coordinates, to generalize Buscher procedure [10, 11, 12, 13] by introducing invariant coordinates defined as line integral of covariant derivatives.

Buscher T-dualization procedure is considered in the scientific literature as definition of T-duality and it is given analytically. But there is an algebraic form of representation of T-duality. This alternative representation is called double space formulation, where T-duality is represented as permutation of coordinates in space spanned by the initial coordinates x^μ and T-dual ones (Lagrange multipliers) y_μ. Double space formulation was appeared first in papers [14, 15, 16, 17, 18], while recently this formalism has been related with $O(D, D)$ transformations [19, 20, 21, 22, 23].

In articles [24, 25] it was shown that for type II superstring theory in pure spinor formulation with constant background fields, Buscher procedure and double space approach give the same results. Here we want to check if the generalized T-dualization procedure works when it is allowed that Ramond-Ramond (R-R) field strength to be linearly dependent on bosonic coordinates.

The motivation why we work with such R-R background field is the assumption (better to say conjecture) presented in the papers [26, 27] that for this choice we will get that anti Poisson bracket of the fermionic coordinates is proportional to the bosonic coordinates x^μ. In [28] that conjecture was not proved. There is also a practical reason for such choice of R-R field strength - model could be treated analytically using the generalized T-duality procedure.

Here we will start deriving the action from the most general one. Integrating out fermionic momenta R-R field strength is coupled with derivatives of bosonic coordinates. Then we will briefly present the results obtained by Buscher T-dualization procedure. Transcribing T-dual transformation laws (relates initial and T-dual coordinates) in terms of the double coordinates, we obtain the T-dual generalized metric and T-dual generalized current. By equating components of the starting and dual generalized metric and generalized current we show the form of the T-dual background fields. Comparing these results with those obtained from Buscher procedure we see that these two approaches are consistent in the case of the coordinate dependent R-R field strength.

2. Type II superstring theory with coordinate dependent R-R field - initial action and its T-dual

In this section we will introduce action for type II superstring in pure spinor formulation [29, 30, 31, 32]. Our choice is the theory that has, except Ramond-Ramond field, all other background fields constant. Ramond-Ramond field is linear function of coordinates x^μ, where the dependence is only infinitesimal. Furthermore, we will assume that RR field is antisymmetric. Both assumptions are of practical nature in order to analytically obtain transformation laws between starting and T-dual theory (T-dual transformation laws).

After that we will give a brief overview of getting the form of the T-dual theory. This theory is both non-commutative and non-associative [28, 33, 34].

At the end we will introduce appropriate notation in the transition toward double space form of T-dual transformation laws.

2.1. Pure spinor formulation of type II superstring

The most general form of the type II superstring action in pure spinor formulation is of the form

$$S = S_0 + V_{SG}, \tag{1}$$

where first term S_0 is free superstring action

$$S_0 = \int_\Sigma d^2\xi \left(\frac{k}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar\theta^\alpha \bar\pi_\alpha \right) + S_\lambda + S_{\bar\lambda}. \tag{2}$$

The integration goes over world-sheet Σ, parametrized by coordinates ξ^m, where $m = 0,1$ $(\xi^0 = \tau,\ \xi^1 = \sigma)$, $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$ and $\partial_\pm = \partial_\tau \pm \partial_\sigma$. The bosonic coordinates $x^\mu,\ \mu = 0,1,...,9$ and fermionic ones θ^α and $\bar\theta^\alpha$, with 16 independent real parameters each $(\alpha = 1,2,...,16)$, make a superspace. Variables π_α and $\bar\pi_\alpha$ are momenta canonically conjugated to the fermionic coordinates. Terms S_λ and $S_{\bar\lambda}$ denote actions for pure spinors. The second term in (1) is the integrated vertex operator for massless type II supergravity and its form is

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar X^N\,, \tag{3}$$

where A_{MN} contains fields that, in general, depend on both bosonic and fermionic coordinates. It is very (mathematically) complicated to work with the action in the general form. Our choice are the following background fields

$$A_{MN} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}) & \bar\Psi^\beta_\mu & 0 \\ 0 & -\Psi^\alpha_\nu & \frac{2}{k}(f^{\alpha\beta} + C^{\alpha\beta}_\rho x^\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

All background fields are constant except Ramon-Ramond field strength. Here $g_{\mu\nu}$ is symmetric tensor, $B_{\mu\nu}$ is Kalb-Ramon antisymmetric tensor, Ψ^α_μ and $\bar\Psi^\alpha_\mu$ are Majorana-Weyl gravitino fields and $\frac{2}{k}(f^{\alpha\beta} + C^{\alpha\beta}_\rho x^\rho) = \frac{2}{k}F^{\alpha\beta}$ is Ramond-Ramond field. Ramond-Ramond field strength is linear function of the bosonic coordinates and it consists of the constant antisymmetric tensors $f^{\alpha\beta}$ and $C^{\alpha\beta}_\rho$. Tensor $C^{\alpha\beta}_\rho$ is supposed to be infinitesimal. This specific choice of the R-R field strength is for the technical reasons - in order to obtain transformation laws that can be easily recast in double space formulation. Since we are only interested in classical analysis, we will not calculate dilaton shift under T-duality transformations. This choice of fields is accompanied with following constraint following from the whole set of the consistency conditions for background fields

$$\gamma^{\mu}_{\alpha\beta} C^{\beta\gamma}_{\mu} = 0, \quad \gamma^{\mu}_{\alpha\beta} C^{\gamma\beta}_{\mu} = 0. \tag{5}$$

The vectors X^M and $\bar{X}^M$ contain partial derivatives of both the fermionic and bosonic coordinates, fermionic momenta and pure spinors. In calculation, in order to simplify it, we will keep just term linear in the fermionic coordinates θ^α and $\bar{\theta}^\alpha$. Consequently, vectors X^M and $\bar{X}^M$ are getting form

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \partial_+ x^\mu \\ \pi_\alpha \\ \frac{1}{2}N^{\mu\nu}_+ \end{pmatrix}, \ \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \partial_- x^\mu \\ \bar{\pi}_\lambda \\ \frac{1}{2}\bar{N}^{\mu\nu}_- \end{pmatrix}, \tag{6}$$

where the components containing pure spinors are

$$N^{\mu\nu}_+ = \frac{1}{2}\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \lambda^\beta, \quad \bar{N}^{\mu\nu}_- = \frac{1}{2}\bar{\omega}_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \bar{\lambda}^\beta. \tag{7}$$

Since pure spinors are decoupled from the rest of the action, we will not consider them in the further analysis.

Taking into account all these assumptions, the action (1) gets the following form

$$\begin{aligned} S &= \kappa \int_\Sigma d^2\xi \, [\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu \\ &+ \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}^\alpha_\mu) \left(F^{-1}(x)\right)_{\alpha\beta} (\partial_-\theta^\beta + \Psi^\beta_\nu \partial_- x^\nu) \Big]. \end{aligned} \tag{8}$$

The fermionic momenta π_α and $\bar{\pi}_\alpha$ are integrated out and we introduced the following notation

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}, \tag{9}$$

$$\begin{aligned} &F^{\alpha\beta}(x) = f^{\alpha\beta} + C^{\alpha\beta}_\mu x^\mu \,, \\ &(F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho x^\rho (f^{-1})_{\beta_1\beta} \,. \end{aligned} \tag{10}$$

2.2. (Generalized) Buscher T-dualization procedure - analytical approach

The details of the Buscher T-dualization procedure of the model presented above is given in papers [28, 33, 34]. Here we will only a brief review of the most important steps and results.

The standard Buscher procedure [5, 9, 11] is carried out only along isometry directions and that is the first task - to detect the isometry directions. Since Ramond-Ramond field is antisymmetric one, the action (8) is invariant to translations along bosonic coordinates. Localizing the noticed symmetry is the next step in procedure. As it is usual, this is made by introducing covariant derivatives instead ordinary ones. The next step is new one in relation to the standard Buscher procedure. Because RR field strength depends on the coordinates linearly, it is also required to introduce *invariant coordinates* in the form of the line integral of covariant derivative. The newly introduced gauge fields add new degrees of freedom. In order to have T-dual theory which is physically equivalent to the initial one, the number of degrees of freedom must be kept. The eliminating of the excess of the degrees of freedom is achieved by introducing Lagrange multipliers. We can also fix starting bosonic coordinates using symmetry which leaves us with theory that is described by only gauge fields and Lagrange multipliers. Finding equations for gauge field and inserting them into the gauge fixed action we are left with T-dual theory. Implementing the described procedure we obtain following T-dual action

$$\begin{aligned} {}^bS &= \frac{\kappa}{2}\int_\Sigma d^2\xi\left[\frac{1}{2}\bar\Theta_-^{\mu\nu}\partial_+y_\mu\partial_-y_\nu+\partial_+\bar\theta^{\alpha b}F^{-1}_{\alpha\beta}(V^{(0)})\partial_-\theta^\beta\right.\\ &+ \left.\partial_+y_\mu{}^b\bar\Psi^{\mu\alpha b}F^{-1}_{\alpha\beta}(V^{(0)})\partial_-\theta^\beta+\partial_+\bar\theta^{\alpha b}F^{-1}_{\alpha\beta}(V^{(0)}){}^b\Psi^{\nu\beta}\partial_-y_\nu\right]. \end{aligned} \quad (11)$$

Let us notice that y_μ is a T-dual coordinate, left superscript ^{b} denotes bosonic T-dualization and V^0 is defined as

$$\begin{aligned} &\Delta V^{(0)\rho} = \\ &= \frac{1}{2}\int_P d\xi^+\breve\Theta_-^{\rho_1\rho}\left[\partial_+y_{\rho_1}-\partial_+\bar\theta^\alpha(f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right]\\ &- \frac{1}{2}\int_P d\xi^-\breve\Theta_-^{\rho\rho_1}\left[\partial_-y_{\rho_1}+\bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \end{aligned} \quad (12)$$

The meaning of the tensors in the above expression: $\bar\Theta_-^{\mu\nu}$ is inverse tensor of

$$\begin{aligned} \bar\Pi_{\pm\mu\nu} &= \Pi_{\pm\mu\nu}+\frac{1}{2}\bar\Psi^\alpha_\mu\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\nu\\ &= \breve\Pi_{\pm\mu\nu}-\frac{1}{2}\bar\Psi^\alpha_\mu(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_\nu \end{aligned} \quad (13)$$

defined as

$$\bar{\Theta}^{\mu\nu}_{\mp}\bar{\Pi}_{\pm\nu\rho} = \delta^\mu_\rho, \tag{14}$$

where

$$\bar{\Theta}^{\mu\nu}_{\mp} = \breve{\Theta}^{\mu\nu}_{\mp} + \frac{1}{2}\breve{\Theta}^{\mu\mu_1}_{\mp}\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho V^{(0)\rho}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\breve{\Theta}^{\nu_1\nu}_{\mp}, \tag{15}$$

$$\breve{\Theta}^{\mu\nu}_{\mp}\breve{\Pi}_{\pm\nu\rho} = \delta^\mu_\rho, \qquad \breve{\Theta}^{\mu\nu}_{\mp} = \Theta^{\mu\nu}_{\mp} - \frac{1}{2}\Theta^{\mu\mu_1}_{\mp}\bar{\Psi}^\alpha_{\mu_1}(\bar{f}^{-1})_{\alpha\beta}\Psi^\beta_{\nu_1}\Theta^{\nu_1\nu}_{\mp}, \tag{16}$$

$$\bar{f}^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_{-}\bar{\Psi}^\beta_\nu, \tag{17}$$

$$\Theta^{\mu\nu}_{\mp}\Pi_{\pm\mu\rho} = \delta^\mu_\rho, \qquad \Theta_{\mp} = -4(G_E^{-1}\Pi_{\mp}G^{-1})^{\mu\nu}, \tag{18}$$

$$G_{E\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \tag{19}$$

$$\Pi_{+\mu\nu} = -\Pi_{-\nu\mu}, \quad \breve{\Pi}_{+\mu\nu} = -\breve{\Pi}_{-\nu\mu}, \quad \bar{\Pi}_{+\mu\nu} = -\bar{\Pi}_{-\nu\mu}, \tag{20}$$

$$\Theta^{\mu\nu}_{+} = -\Theta^{\nu\mu}_{-}, \quad \breve{\Theta}^{\mu\nu}_{+} = -\breve{\Theta}^{\nu\mu}_{-}, \quad \bar{\Theta}^{\mu\nu}_{+} = -\bar{\Theta}^{\nu\mu}_{-}. \tag{21}$$

The relation between ${}^bF^{-1}_{\alpha\beta}(V^{(0)})$ and $F^{-1}_{\alpha\beta}(x)$, is of the form

$${}^bF^{-1}_{\alpha\beta}(V^{(0)}) = F^{-1}_{\alpha\beta}(V^{(0)}) - \frac{1}{2}F^{-1}_{\alpha\alpha_1}(V^{(0)})\Psi^{\alpha_1}_\mu\bar{\Theta}^{\mu\nu}_{-}\bar{\Psi}^{\beta_1}_\nu F^{-1}_{\beta_1\beta}(V^{(0)}). \tag{22}$$

The expressions for T-dual gravitino fields, ${}^b\bar{\Psi}^{\mu\alpha}$ and ${}^b\Psi^{\nu\beta}$, are the following

$${}^b\bar{\Psi}^{\mu\alpha} = \frac{1}{2}\Theta^{\mu\nu}_{-}\bar{\Psi}^\alpha_\mu, \qquad {}^b\Psi^{\nu\beta} = -\frac{1}{2}\Psi^\beta_\mu\Theta^{\mu\nu}_{-}. \tag{23}$$

One of the main results are T-dual transformation laws connecting starting and T-dual coordinates

$$\bar{\Pi}_{+\mu\nu}\partial_- x^\nu = -\frac{1}{2}\partial_- y_\mu - \frac{1}{2}\bar{\Psi}^\alpha_\mu F^{-1}_{\alpha\beta}(x)\partial_-\theta^\beta - \beta^+_\mu(x), \tag{24}$$

$$\bar{\Pi}_{+\mu\nu}\partial_+ x^\nu = \frac{1}{2}\partial_+ y_\nu - \frac{1}{2}\partial_+\bar{\theta}^\alpha F^{-1}_{\alpha\beta}(x)\Psi^\beta_\nu - \beta^-_\nu(x), \tag{25}$$

where β^+_μ and β^-_μ are β-functions obtained varying gauge fixed action with respect to gauge fields

$$\beta^+_\mu(x) = - \frac{1}{2}(\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1})(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}(\partial_-\theta^\beta + \partial_- x^{\nu_2}\Psi^\beta_{\nu_2}), \tag{26}$$

$$\beta^-_\mu(x) = - \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1})(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}(\theta^\beta + x^{\nu_2}\Psi^\beta_{\nu_2}). \tag{27}$$

Now we have all ingredients for double formalism and proceed further.

3. Double space approach to T-dualization

T-dualization is a kind of transformation. It maps initial space spanned by x^μ coordinates to the T-dual space spanned by y_μ coordinates. But if we unite T-dual coordinates y_μ and initial space coordinates x^μ, we get **double space**. In double space T-dualization is a symmetry transformation represented as permutation of . Here we present the results from [24, 25], where same model was examined but with constant background fields.

3.1. T-dual transformation laws in double space representation

For practical reasons we have to introduce appropriate notation

$$\Psi^\alpha_\mu = \Psi^\alpha_{+\mu}, \quad \bar{\Psi}^\alpha_\mu = \Psi^\alpha_{-\mu}, \quad \theta^\alpha = \theta^\alpha_+, \quad \bar{\theta}^\alpha = \theta^\alpha_-, \tag{28}$$

$$\left(F^{-1}(x)\right)_{\alpha\beta} = \left(F^{-1}_+(x)\right)_{\alpha\beta}, \quad \left(F^{-1}(x)\right)_{\beta\alpha} = \left(F^{-1}_-(x)\right)_{\alpha\beta}, \tag{29}$$

$$\left(F^{-1}_+(x)\right)_{\alpha\beta} = -\left(F^{-1}_-(x)\right)_{\alpha\beta}, \tag{30}$$

$$(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} = C_{+\mu\alpha\beta}, \quad (f^{-1})_{\beta\beta_1} C^{\beta_1\alpha_1}_\mu (f^{-1})_{\alpha_1\alpha} = C_{-\mu\alpha\beta}, \tag{31}$$

$$C_{+\mu\alpha\beta} = -C_{-\mu\alpha\beta}. \tag{32}$$

Using above notation we rewrite the T-dual transformation law, (24) and (25), in the following way

$$\begin{aligned} \partial_\mp x^\nu &= -\frac{1}{2}\widehat{\Theta}^{\nu\mu}_\mp \partial_\mp y_\mu \\ &- \frac{1}{2}\widehat{\Theta}^{\nu\mu}_\mp \left[\Psi^\alpha_{\mp\mu} F^{-1}_{\pm\alpha\beta}(V^{(0)}) - (\theta^\alpha_\mp + V^{(0)\nu_1}\Psi^\alpha_{\mp\nu_1}) C_{\pm\mu\alpha\beta}\right] \partial_\mp \theta^\beta_\pm, \end{aligned} \tag{33}$$

$$\partial_\mp y_\mu = -2\widehat{\Pi}_{\pm\mu\nu}\partial_\mp x^\nu - \left[\Psi^\alpha_{\mp\mu} F^{-1}_{\pm\alpha\beta}(x) - (\theta^\alpha_\mp + x^{\nu_1}\Psi^\alpha_{\mp\nu_1}) C_{\pm\mu\alpha\beta}\right] \partial_\mp \theta^\beta_\pm, \tag{34}$$

where $\beta^\pm_\mu$ functions are in the expanded form divided into two parts - the first one contains partial derivative of bosonic coordinates and the other one contains partial derivative of fermionic coordinates. The first part is added to the tensors $\widehat{\Pi}_{\pm\mu\nu}$ and $\widehat{\Theta}^{\nu\mu}_\mp$

$$\widehat{\Pi}_{\pm\mu\nu} = \bar{\Pi}_{\pm\mu\nu} - \frac{1}{2}(\theta^\alpha_\mp + x^{\nu_1}\Psi^\alpha_{\mp\nu_1}) C_{\pm\mu\alpha\beta}\Psi^\beta_{\pm\nu}, \tag{35}$$

$$\widehat{\Theta}^{\nu\mu}_\mp = \bar{\Theta}^{\nu\mu_1}_\mp \left[\delta^\mu_{\mu_1} + \frac{1}{2}(\theta^\alpha_\mp + V^{(0)\nu_1}\Psi^\alpha_{\mp\nu_1}) C_{\pm\mu_1\alpha\beta}\Psi^\beta_{\pm\nu_2}\breve{\Theta}^{\nu_2\mu}_\mp\right], \tag{36}$$

$$\widehat{\Pi}_{\pm\mu\nu}\widehat{\Theta}^{\nu\rho}_\mp = \delta^\rho_\mu\,. \tag{37}$$

These two tensors can be decomposed as it follows

$$\widehat{\Pi}_{\pm\mu\nu} = \widehat{B}_{\mu\nu} \pm \frac{1}{2}\widehat{G}_{\mu\nu}, \quad \widehat{\Theta}^{\mu\nu}_{\mp} = -4(\widehat{G}^{-1}_E \widehat{\Pi}_{\mp}\widehat{G}^{-1})^{\mu\nu}, \tag{38}$$

$$\widehat{G}_{E\mu\nu} = \widehat{G}_{\mu\nu} - 4\widehat{B}_{\mu\mu_1}\widehat{G}^{\mu_1\nu_1}\widehat{B}_{\nu_1\nu}, \tag{39}$$

$$\widehat{\Theta}^{\nu\mu}_{\pm} = -4\widehat{G}^{\nu\nu_1}_E \widehat{B}_{\nu_1\mu_1}\widehat{G}^{\mu_1\mu} \mp 2(\widehat{G}^{-1}_E)^{\nu\mu}. \tag{40}$$

Using these decompositions allow us to rewrite transformation laws as

$$\begin{aligned} \pm \quad & \partial_\pm x^\mu = (\widehat{G}^{-1})^{\mu\nu}\partial_\pm y_\nu + 2(\widehat{G}^{-1})^{\mu\nu_1}\widehat{B}_{\nu_1\nu}\partial_\pm x^\nu \\ + \quad & (\widehat{G}^{-1})^{\mu\nu}\left[\Psi^\alpha_{\pm\nu}F^{-1}_{\mp\alpha\beta}(x) - (\theta^\alpha_\pm + x^{\nu_1}\Psi^\alpha_{\pm\nu_1})C_{\mp\nu\alpha\beta}\right]\partial_\pm\theta^\beta_\mp, \end{aligned} \tag{41}$$

$$\begin{aligned} \pm \quad & \partial_\pm y_\mu = \widehat{G}_{E\mu\nu}\partial_\pm x^\nu - 2\widehat{B}_{\mu\mu_1}\widehat{G}^{\mu_1\nu}\partial_\pm y_\nu \\ + \quad & \frac{1}{2}\widehat{G}_{E\mu\nu}\widehat{\Theta}^{\nu\mu_1}_{\pm}\left[\Psi^\alpha_{\pm\mu}F^{-1}_{\mp\alpha\beta}(V^{(0)}) - (\theta^\alpha_\pm + V^{(0)\nu_1}\Psi^\alpha_{\pm\nu_1})C_{\mp\mu_1\alpha\beta}\right]\partial_\pm\theta^\beta_\mp. \end{aligned} \tag{42}$$

Let us note that all tensors in equation (41) are functions of the initial bosonic coordinates x^μ. Further, all tensors in (42) are functions of $V^{(0)}$ which has been defined as line integral (12).

As we said in the introductory part of this section, uniting initial and T-dual space we get the double space spanned by the coordinates

$$Z^A = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix} \tag{43}$$

In double space the T-dual transformation laws (41) and (42) can be written in the very elegant form

$$\pm\Omega_{MN}\partial_\pm Z^N = \breve{\mathcal{H}}_{MN}\partial_\pm Z^N + \breve{J}_{\pm M}, \tag{44}$$

where generalized metric is given as

$$\breve{\mathcal{H}}_{MN} = \begin{pmatrix} \widehat{G}_{E\mu\nu}(V) & -2\widehat{B}_{\mu\mu_1}(\widehat{G}^{-1})^{\mu_1\nu}(V) \\ 2(\widehat{G}^{-1})^{\mu\nu_1}\widehat{B}_{\nu_1\nu}(x) & (\widehat{G}^{-1})^{\mu\nu}(x) \end{pmatrix}. \tag{45}$$

The double current is defined as

$$\begin{aligned} \breve{J}_{\pm M} &= \begin{pmatrix} \frac{1}{2}\widehat{G}_{\mu\nu_1}\widehat{\Theta}^{\nu_1\nu}_{\pm}(V) \\ (\widehat{G}^{-1})^{\mu\nu}(x) \end{pmatrix} J_{\pm\nu}, \\ J_{\pm\nu} &= \left[\Psi^\alpha_{\pm\nu}F^{-1}_{\mp\alpha\beta}(x) - (\theta^\alpha_\pm + x^{\nu_1}\Psi^\alpha_{\pm\nu_1})C_{\mp\nu\alpha\beta}\right]\partial_\pm\theta^\beta_\mp. \end{aligned} \tag{46}$$

The upper components all depend on variable $V^{(0)}$, while lower components all depend on x.

The matrix

$$\Omega_{MN} = \begin{pmatrix} 0 & I_D \\ I_D & 0 \end{pmatrix}, \tag{47}$$

is invariant $SO(D,D)$ metric where I_D denotes unity matrix in D dimensions. Further, it satisfies the following relations

$$\breve{\mathcal{H}}^T \Omega \breve{\mathcal{H}} = \Omega, \quad (\Omega\breve{\mathcal{H}})^2 = I, \quad \Omega^2 = I, \quad \det(\breve{\mathcal{H}}) = 1, \tag{48}$$

which means that $\breve{\mathcal{H}} \in SO(D,D)$ [14, 19].

3.2. T-duality in double space

T-dualization in double space is represented as permutation of the initial and T-dual coordinates. So, first we have to introduce the permutation matrix

$$T^M{}_N = \begin{pmatrix} 0 & I_D \\ I_D & 0 \end{pmatrix}. \tag{49}$$

T-dual double coordinate is of the form

$${}^b Z^M = T^M{}_N Z^N, \tag{50}$$

and its transformation law is of the same form as for the initial dual coordinate Z^A (44). So, we have

$$\pm\Omega_{MN}\partial_\pm {}^b Z^N = {}^b\breve{\mathcal{H}}_{MN}\partial_\pm {}^b Z^N + {}^b J_{\pm M}. \tag{51}$$

From this we derive how the T-dualized generalized metric and double current are related with the initial ones

$${}^b\breve{\mathcal{H}}_{MN} = T_M{}^P \breve{\mathcal{H}}_{PQ} T^Q{}_N, \quad {}^b\breve{J}_{\pm M} = T_M{}^N \breve{J}_{\pm N}. \tag{52}$$

The first relation produces

$$\begin{aligned} {}^b\breve{\mathcal{H}}_{MN} &= \begin{pmatrix} {}^b\widehat{G}_E^{\mu\nu}(V) & -2\, {}^b\widehat{B}^{\mu\mu_1} ({}^b\widehat{G}^{-1})_{\mu_1\nu}(V) \\ 2\, ({}^b\widehat{G}^{-1})_{\mu\nu_1} {}^b\widehat{B}^{\nu_1\nu}(x) & ({}^b\widehat{G}^{-1})_{\mu\nu}(x) \end{pmatrix} \\ &= \begin{pmatrix} (\widehat{G}^{-1})^{\mu\nu}(x) & 2(\widehat{G}^{-1})^{\mu\nu_1}\widehat{B}_{\nu_1\nu}(x) \\ -2\widehat{B}_{\mu\mu_1}(\widehat{G}^{-1})^{\mu_1\nu}(V) & \widehat{G}_{E\mu\nu}(V) \end{pmatrix}. \end{aligned} \tag{53}$$

Let us note that variables $V^{(0)}$ and x also exchange places. Equating components (2, 2) and (2, 1)

$${}^b\widehat{G}_{\mu\nu}(x) = (\widehat{G}_E^{-1})_{\mu\nu}(V), \tag{54}$$

$${}^b\widehat{B}^{\mu\nu}(x) = -(\widehat{G}_E^{-1})^{\mu\nu_1}\widehat{B}_{\nu_1\mu_1}(\widehat{G}^{-1})^{\mu_1\nu}(V), \tag{55}$$

we derive the form of the T-dual field ${}^b\widehat{\Pi}_{\pm\mu\nu}(x)$

$$ {}^b\widehat{\Pi}_{\pm\mu\nu}(x) = {}^b\widehat{B}_{\mu\nu}(x) \pm \frac{1}{2}{}^b\widehat{G}_{\mu\nu}(x) = \frac{1}{4}\widehat{\Theta}_{\mp\mu\nu}. \tag{56} $$

This is the same result as one obtained in the Buscher procedure. In the same way we obtain the form of the T-dual current

$$ {}^b\breve{J}_{\pm M} = \begin{pmatrix} \frac{1}{2}{}^b\widehat{G}_{\mu\nu_1}{}^b\widehat{\Theta}_{\pm}^{\nu_1\nu}(V) \\ ({}^b\widehat{G}^{-1})^{\mu\nu}(x) \end{pmatrix} J_{\pm\nu} = \begin{pmatrix} (\widehat{G}^{-1})^{\mu\nu}(x) \\ \frac{1}{2}\widehat{G}_{\mu\nu_1}\widehat{\Theta}_{\pm}^{\nu_1\nu}(V) \end{pmatrix} J_{\pm\nu}, \tag{57} $$

where T-dual current has the same factor $J_{\pm\nu}$ while vector components are switched.

Comparing these results to ones obtained in the Buscher procedure performed in [25], we notice that while generalized metric, double current and double space transformation laws have same form, but with some modifications stemming from the fact that starting theory has coordinate dependent RR field.

4. Concluding remarks

The goal of the investigation presented in this paper was to investigate the double space method of T-dualization and compare the results with those obtained using basic approach - using generalized Buscher T-dualization procedure. The object of the analysis was type II superstring theory with antisymmetric linearly coordinate dependent RR field strength. Buscher approach was already carried out in the in the papers [28, 33, 34].

First we justified the choice of the model and specific background fields. All constant background fields, except Ramond-Ramond field, is the choice which is in accordance with consistency conditions. RR field strength was chosen to have infinitesimal linear dependence on bosonic coordinates x^μ. Furthermore, we put RR field strength under one more condition - it is totally antisymmetric. All terms non-linear in fermionic coordinates have been neglected and fermionic momenta, π_α and $\bar{\pi}_\alpha$, are integrated out of the action. These assumptions were necessary in order to obtain simple transformation laws between starting and dual coordinates (for case where RR field is not antisymmetric see [33]). After this we briefly presented, T-dualization within the generalized T-dualization procedure.

Section 3 contains the presentation and application of double space T-dualization mechanism. First we "repack" T-dual transformation laws in the more appropriate form. Combining space of the initial theory, spanned by x^μ, and space of T-dual theory, spanned with y_μ, we made double space, which is spanned by coordinates $Z^M = (x^\mu, y_\mu)$. Rewriting T-dual transformation laws using double space, we introduced two new objects, generalized metric $\breve{\mathcal{H}}_{MN}$ and double current $\breve{J}_{\pm M}$. Their components are expressed in terms of the improved Kalb-Ramond field, improved metric tensor and improved effective metric. "Improved" means that they contain additional terms bilinear in NS-R background Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$. Furthermore, these components are not constant - upper row depends on coordinate x^μ and lower row on its T-dual V^μ.

As we can see T-duality in the double space is given by simple permutation of coordinates. From demand that double space coordinates Z^M and their T-dual ${}^bZ^M = T^M{}_N Z^N$ have the transformation law of the same form, we are able to find T-dual generalized metric ${}^b\breve{\mathcal{H}}_{MN}$ and T-dual double current ${}^b\breve{J}_{\pm M}$. Since the form of the T-dual generalized metric is the same as the form of the starting one, comparing corresponding components, we can derive expressions for T-dual background fields as functions of starting fields. Doing same analysis for T-dual double current, we obtain relations that connect its components to background fields of starting theory.

Comparing results obtained for T-dual fields using double space coordinate permutation with ones obtained with Buscher procedure, it is evident that both methods produce the same results. Additionally, it should be noted that comparing results from this paper with results from [25], where all background fields were constant, we notice that double space transformation laws have the same form but individual components of generalized metric and double current are now coordinate dependents.

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