

Gravitational reheating and bounds in oscillating backgrounds ^{*}

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ABSTRACT

In this article, we determine the reheating temperature in cosmological scenarios where heavy scalar particles are generated gravitationally through a conformally coupled interaction between a massive scalar field and the Ricci scalar. We derive formulas to calculate reheating temperatures based on the mass and decay rate of the produced particles, setting limits on their viable values. Lastly, we examine the gravitational production of dark matter within this framework.

1. Introduction

According to cosmological inflation theory, the Universe underwent a brief period of rapid expansion driven by a scalar field named *inflaton* [1]. As this inflationary phase concludes, the Universe must transition from accelerated expansion to a hot, radiation-dominated phase, aligning with Big Bang Nucleosynthesis (BBN) and leading to the observable Universe. This vital transition happens through a process called reheating [2].

In this framework, gravitational reheating [3, 4] offers a unique approach where a quantum field coupled to gravity generates massive particles. These particles decay into Standard Model (SM) particles and other forms of matter and radiation, heating the Universe. Unlike conventional reheating, which involves specific couplings between the inflaton and other fields, gravitational reheating operates through the spacetime dynamics itself, making it independent of particular particle interactions.

The effectiveness of gravitational reheating heavily depends on the behavior of the inflaton field and its potential. A crucial factor is how quickly the inflaton's energy density decreases; if it declines slower than radiation, it could re-dominate the Universe's energy budget, creating conflicts with the concordance cosmological model. To avoid this, it's often required

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that the inflaton's energy density decreases faster than radiation's, imposing conditions on the effective equation of state (EoS) parameter, w_{eff} , of the inflaton field. Specifically, successful gravitational reheating requires $w_{\text{eff}} > 1/3$, which implies certain conditions on the inflaton potential, such as $n > 2$ for a potential of the form φ^{2n} .

In this work, our main objective is to derive analytic expressions for the reheating temperature within the framework of gravitational particle production, utilizing the so-called Bogoliubov approach [5]. A key element in determining these expressions is the energy density of the produced particles at the end of inflation, calculated analytically using the Stokes phenomenon [6]. With this energy density, we derive a formula for the reheating temperature as a function of the decay rate and the mass m_χ of the produced particles, identifying values of m_χ and the corresponding range of possible reheating temperatures.

Additionally, in scenarios without strong couplings between the inflaton and standard particles, gravitational reheating can generate a non-thermal particle spectrum, including potential dark matter candidates. This mechanism is particularly compelling in models where dark matter interacts minimally with ordinary matter. During gravitational reheating, dark matter production occurs efficiently through gravitational interactions alone, providing a universal production mechanism that does not require direct couplings to the inflaton. With this in mind, we examine the gravitational production of dark matter within the gravitational reheating framework. We specifically consider two different scalar fields conformally coupled to gravity, producing two types of particles with distinct masses. One particle type decays into lighter particles to reheat the Universe, while the other remains stable, serving as a dark matter candidate. In this work, we determine the particle masses that provide both a viable reheating temperature and a present-day cold dark matter density, contributing to a more complete understanding of the Universe's dark matter and energy composition.

Throughout the manuscript we use natural units, i.e., $\hbar = c = k_B = 1$, and the reduced Planck's mass is denoted by $M_{\text{pl}} \equiv \frac{1}{\sqrt{8\pi G}} \cong 2.44 \times 10^{18}$ GeV.

2. Gravitational Particle Production

In the context of a flat Friedman-Lemaître-Robertson-Walker (FLRW) spacetime, with $a(t)$ as the Universe's scale factor, we define $\rho_B(t)$ as the background energy density and $\langle \rho(t) \rangle$ as the energy density of the gravitationally produced, heavy scalar particles that are conformally coupled. Assuming that near the potential minimum (taken as $\varphi_{\text{min}} = 0$ for simplicity), the potential follows a φ^{2n} form, as in models such as the one studied in [8]

$$V_n(\varphi) = \lambda M_{\text{pl}}^4 \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\text{pl}}}} \right)^{2n}, \quad (1)$$

where the value of the dimensionless constant $\lambda \sim 3\pi^2(1 - n_s)^2 10^{-9}$ — $n_s \sim 0.96$ is the spectral index of scalar perturbations— is obtained from the power spectrum of scalar perturbations. Using the virial theorem, one can conclude that, when the inflaton oscillates, the effective EoS parameter is given by $w_{\text{eff}} = (n - 1)/(n + 1)$ [9]. Since after inflation, the energy density of the produced particles (or that of their decay products, e.g. light relativistic particles) eventually has to dominate that of the inflaton for the Universe to become reheated, we must demand that the energy density of the inflaton decreases faster than that of radiation. This requires $w_{\text{eff}} > 1/3 \implies n > 2$. Otherwise, if the energy density of the inflaton decreases more slowly than that of radiation, it could dominate again during this period, which is incompatible with the concordance model. Therefore, potentials appearing in Starobinsky ($n = 1$) or Higgs inflation [10], cannot contain gravitational reheating as a mechanism to reheat the universe.

Remark 2.1: *It is crucial to understand that when the potential behaves as a quartic one, specifically φ^4 , i.e., $n = 2$, the model remains viable only if the decay of the massive particles occurs well after the inflaton's energy density has dominated. In this scenario, both energy densities decrease at the same rate, proportional to $a^{-4}(t)$, but the inflaton's energy density will not dominate in the future.*

Remark 2.2: *It is also possible to realize the cases $n = 1$ and $n = 2$ by assuming that the inflaton decays into relativistic particles of the Standard Model (SM) [11, 12]. However, this implies that the quantum field is coupled with the inflaton field, which does not precisely align with the concept of "gravitational particle production" [4, 13, 3, 14], where the quantum field interacts with the Ricci scalar. Alternatively, one might consider that reheating effectively occurs due to the coupling between the inflaton field and a quantum field responsible for producing SM particles, while also accounting for another massive spectator quantum field coupled to gravity, which could serve as a candidate for gravitationally producing dark matter [15, 7, 16, 17].*

During the inflaton's oscillations, which occur shortly after inflation ends and before the produced particles decay, the evolution of its energy density, averaged over time, can be described as follows (with "END" marking the end of inflation):

$$\rho_B(t) = \rho_{\text{B,END}} \left(\frac{a_{\text{END}}}{a(t)} \right)^{3(1+w_{\text{eff}})} = \rho_{\text{B,END}} \left(\frac{a_{\text{END}}}{a(t)} \right)^{\frac{6n}{n+1}}. \quad (2)$$

In addressing the gravitational production of particles, we consider the Lagrangian density for a real massive scalar field that interacts with the

Ricci scalar [18]:

$$L = \frac{1}{2} \sqrt{|g|} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m_\chi^2 \phi^2 - \xi R \phi^2), \quad (3)$$

where m_χ is the mass of the field, R is the Ricci scalar and ξ is the dimensionless coupling constant. The Euler-Lagrange equation leads to the following dynamical equation

$$(\nabla^\mu \nabla_\mu + m_\chi^2 + \xi R) \phi = 0. \quad (4)$$

To facilitate the procedure, we consider the conformally coupled case $\xi = 1/6$, and thus, the Lagrangian becomes

$$L = \frac{a^2}{2} \left\{ (\phi')^2 - (\nabla \phi)^2 - a^2 \left[m_\chi^2 + \frac{R}{6} \right] \phi^2 \right\}, \quad (5)$$

where the prime denotes the derivative with respect to the conformal time η and ∇ is the gradient operator. To simplify this expression we perform the change of variable $\phi = \chi/a$, obtaining

$$L = \frac{1}{2} \left\{ (\chi')^2 - (\nabla \chi)^2 - a^2 m_\chi^2 \chi^2 \right\} - \frac{1}{2} \frac{d}{d\eta} \left(\chi^2 \frac{a'}{a} \right), \quad (6)$$

and because the Lagrangian (6) includes a total derivative, it is dynamically equivalent to

$$\bar{L} = \frac{1}{2} \left\{ (\chi')^2 - (\nabla \chi)^2 - a^2 m_\chi^2 \chi^2 \right\}, \quad (7)$$

thus, applying the Euler-Lagrange equations, we obtain its equation of motion

$$\chi'' - \Delta \chi + a^2 m_\chi^2 \chi = 0, \quad (8)$$

where in Fourier space takes the form

$$\chi_k'' + \omega_k^2(\eta) \chi_k = 0, \quad (9)$$

being $\omega_k(\eta) = \sqrt{k^2 + m_\chi^2 a^2(\eta)}$ the frequency of the k -mode. Next, we will expand the modes in the following way,

$$\chi_k(\eta) = \alpha_k(\eta) \phi_{k,+}(\eta) + \beta_k(\eta) \phi_{k,-}(\eta), \quad (10)$$

where $\alpha_k(\eta)$ and $\beta_k(\eta)$ are the time-dependent Bogoliubov coefficients and we have introduced the positive (+) and negative (-) frequency *adiabatic modes*

$$\phi_{k,\pm}(\eta) = \frac{e^{\mp i \int_{\eta_i}^{\eta} \omega_k(\tau) d\tau}}{\sqrt{2\omega_k(\eta)}}. \quad (11)$$

Imposing that the modes satisfy the condition

$$\chi'_k(\eta) = -i\omega_k(\eta) (\alpha_k(\eta)\phi_{k,+}(\eta) - \beta_k(\eta)\phi_{k,-}(\eta)), \quad (12)$$

one can show that the Bogoliubov coefficients are subjected to the differential equation [19]

$$\begin{cases} \dot{\alpha}_k(t) &= \frac{\dot{\omega}_k(t)}{2\omega_k(t)} e^{-2i \int^t \frac{\omega_k(t)}{a(t)} dt} \beta_k(t) \\ \dot{\beta}_k(t) &= \frac{\dot{\omega}_k(t)}{2\omega_k(t)} e^{2i \int^t \frac{\omega_k(t)}{a(t)} dt} \alpha_k(t), \end{cases} \quad (13)$$

where an overdot denotes the derivative with respect to the cosmic time, and the following relation $|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1$, holds. It is crucial to recognize that the β -Bogoliubov coefficients encapsulate both vacuum polarization effects and particle production. Shortly after the initiation of the oscillations, the polarization effects become negligible. This implies that when they stabilize at a value denoted by β_k , the coefficients only reflect the contribution of the produced particles [20], and its energy density, after the stabilization of the β -Bogoliubov coefficients, is given by [18]:

$$\langle \rho(t) \rangle = \frac{1}{2\pi^2 a^4(t)} \int_0^\infty k^2 \omega_k(t) |\beta_k|^2 dk \cong \frac{m_\chi}{2\pi^2 a^3(t)} \int_0^\infty k^2 |\beta_k|^2 dk. \quad (14)$$

Once we comprehend the evolution of these energy densities prior to the decay of the heavy particles into lighter ones, two distinct scenarios emerge: nearly instantaneous decay and non-instantaneous decay. As a result, we will examine both situations with careful attention to detail.

2.1. Instantaneous decay

Let Γ be the decay rate of the heavy massive particles, and it is worth noting that the decay process ends when Γ is of the same order as the Hubble rate. Denoting $\rho_{\text{B,dec}}$ and $\langle \rho_{\text{dec}} \rangle$ as the energy density of the background and that of the produced particles at the end of the decay, respectively, after the decay, which we will assume nearly instantaneous and without a substantial drop of energy, the corresponding energy densities evolve as:

$$\rho_{\text{B}}(t) = \rho_{\text{B,dec}} \left(\frac{a_{\text{dec}}}{a(t)} \right)^{\frac{6n}{n+1}} \quad \text{and} \quad \langle \rho(t) \rangle = \langle \rho_{\text{dec}} \rangle \left(\frac{a_{\text{dec}}}{a(t)} \right)^4, \quad (15)$$

because, after the decay, the particles are currently relativistic.

Hence, given the virtually instantaneous nature of the thermalization process, the universe undergoes reheating at the conclusion of the inflaton's domination, denoted by the sub-index ‘‘end’’, that is, when both energy densities are of the same order. This leads to the relation:

$$\left(\frac{a_{\text{dec}}}{a_{\text{end}}} \right)^{\frac{2(n-2)}{n+1}} = \frac{\langle \rho_{\text{dec}} \rangle}{\rho_{\text{B,dec}}} \implies \langle \rho_{\text{end}} \rangle = \langle \rho_{\text{dec}} \rangle^{\frac{3n}{n-2}} \rho_{\text{B,dec}}^{-\frac{2(n+1)}{n-2}}. \quad (16)$$

Therefore, from the Stefan-Boltzmann law, the reheating temperature has the following expression:

$$T_{\text{reh}}(n) \equiv \left(\frac{30}{\pi^2 g_{\text{reh}}} \right)^{1/4} \langle \rho_{\text{end}} \rangle^{1/4} = \left(\frac{30}{\pi^2 g_{\text{reh}}} \right)^{1/4} \langle \rho_{\text{dec}} \rangle^{1/4} \left(\frac{\langle \rho_{\text{dec}} \rangle}{\rho_{\text{B,dec}}} \right)^{\frac{n+1}{2(n-2)}}, \quad (17)$$

where $g_{\text{reh}} = 106.75$, is the effective number of degrees of freedom for the Standard Model. At this point, we can refine this formula by taking into account the evolution of the associated energy densities prior to decay. They can be represented by the following expressions:

$$\rho_{\text{B}}(t) = 3H_{\text{END}}^2 M_{\text{pl}}^2 \left(\frac{a_{\text{END}}}{a(t)} \right)^{\frac{6n}{n+1}} \quad (18)$$

$$\langle \rho(t) \rangle = \langle \rho_{\text{END}} \rangle \left(\frac{a_{\text{END}}}{a(t)} \right)^3. \quad (19)$$

Note that, one can calculate analytically $\rho_{\text{B,END}}$, because since the end of inflation occurs when the slow roll parameter $\epsilon = \frac{M_{\text{pl}}^2}{2} \left(\frac{dV}{d\varphi} / V \right)^2$ is equal to 1, one can calculate analytically $\rho_{\text{B,END}} = \frac{3}{2} V(\varphi_{\text{END}}) \equiv \frac{3}{2} V_{\text{END}}$. Then, when the heavy particles have completely decayed, which occurs when $H \sim \Gamma$, the semi-classical Friedmann equation $H^2 = \frac{1}{3M_{\text{pl}}^2} (\rho_{\text{B}} + \langle \rho \rangle)$, becomes,

$$3\Gamma^2 M_{\text{pl}}^2 = \rho_{\text{B,END}} x^{\frac{2n}{n+1}} + \langle \rho_{\text{END}} \rangle x, \quad (20)$$

where we have introduced the notation $x = (a_{\text{END}}/a_{\text{dec}})^3$. The solution has the form $x = F_n(\Theta, \bar{\Theta})$, where F_n is a function which depends on n and the parameters, $\Theta = \langle \rho_{\text{END}} \rangle / \rho_{\text{B,END}}$, named *heating efficiency*, and $\bar{\Theta} = 3\Gamma^2 M_{\text{pl}}^2 / \rho_{\text{B,END}}$. For example, when $n = \infty$, i.e. when $w_{\text{eff}} = 1$, ones has

$$F_{\infty}(\Theta, \bar{\Theta}) = \frac{1}{2} \left(\sqrt{\Theta^2 + 4\bar{\Theta}} - \Theta \right), \quad (21)$$

and for $n = 3$ we obtain $x^{3/2} + \Theta x - \bar{\Theta} = 0$, which is a cubic equation when one introduces the variable $x = z^2$. This equation can be solved using the Cardano's formulas, obtaining:

$$F_3(\Theta, \bar{\Theta}) = \left[\sqrt[3]{\frac{1}{2} \left(\bar{\Theta} - \frac{2\Theta^3}{27} + \sqrt{\bar{\Theta} \left(\bar{\Theta} - \frac{4\Theta^3}{27} \right)} \right)} + \sqrt[3]{\frac{1}{2} \left(\bar{\Theta} - \frac{2\Theta^3}{27} - \sqrt{\bar{\Theta} \left(\bar{\Theta} - \frac{4\Theta^3}{27} \right)} \right)} - \frac{\Theta}{3} \right]^2, \quad (22)$$

provided that, $\bar{\Theta} - \frac{4\Theta^3}{27} > 0$, which always happens. Effectively, from the equation $x^{3/2} + \Theta x - \bar{\Theta} = 0$ we have the bound $\bar{\Theta} > \Theta x$. On the other hand, since the decay occurs during the domination of the inflaton's energy density, one has $\langle \rho_{\text{dec}} \rangle \leq \rho_{\text{B,dec}}$ which is equivalent to $\Theta^2 \leq x$, then we have:

$$\bar{\Theta} - \frac{4\Theta^3}{27} > \Theta x \left(1 - \frac{4}{27}\right) > 0. \quad (23)$$

Coming back to the reheating formula, we can find the final expression of the reheating temperature as a function of the parameters Θ and $\bar{\Theta}$:

$$\begin{aligned} T_{\text{reh}}(n) &\cong 5 \times 10^{-1} \left(\frac{\Theta^3}{F_n(\Theta, \bar{\Theta})} \right)^{\frac{n}{4(n-2)}} \sqrt{H_{\text{END}} M_{\text{pl}}} \\ &\sim 5 \times 10^{-4} \left(\frac{\Theta^3}{F_n(\Theta, \bar{\Theta})} \right)^{\frac{n}{4(n-2)}} M_{\text{pl}}, \end{aligned} \quad (24)$$

where we have taken, as usual for many inflationary models, $H_{\text{END}} \sim 10^{-6} M_{\text{pl}}$. On the other hand, given that the decay occurs during the domination of the inflaton's energy density, we have the constraints $\Gamma \leq H_{\text{END}}$ and $\langle \rho_{\text{dec}} \rangle \leq \rho_{\text{B,dec}}$, which after some algebra, is equivalent to $\Theta^{\frac{n+1}{n-1}} \leq F_n(\Theta, \bar{\Theta})$. What leads to the following constraint on the parameters Θ and $\bar{\Theta}$:

$$\Theta^{\frac{n+1}{n-1}} \leq F_n(\Theta, \bar{\Theta}) \quad \text{and} \quad \bar{\Theta} \leq 1. \quad (25)$$

Finally, we calculate the maximum reheating temperature, which is obtained when the decay occurs at the end of inflaton's domination, that is, when $\langle \rho_{\text{dec}} \rangle = \rho_{\text{B,dec}}$, which after some algebra, is equivalent to:

$$F_n(\Theta, \bar{\Theta}) = \Theta^{\frac{n+1}{n-1}}, \quad (26)$$

and thus, the maximum reheating temperature has the following expression as a function of the parameter Θ :

$$T_{\text{reh}}^{\text{max}}(n) \cong 5 \times 10^{-1} \Theta^{\frac{n}{2(n-1)}} \sqrt{H_{\text{END}} M_{\text{pl}}} \sim 5 \times 10^{-4} \Theta^{\frac{n}{2(n-1)}} M_{\text{pl}}. \quad (27)$$

Note that the maximum reheating temperature increases as n increases because, with higher n , the energy density of the produced particles overtakes that of the inflaton more quickly. Therefore, the maximum value is obtained for potentials that, when the inflaton oscillates, the universe enters in a stiff matter era, i.e., $w_{\text{eff}} = 1$. In this situation, the maximum reheating temperature is given by,

$$T_{\text{reh}}^{\text{max}}(\infty) \cong 5 \times 10^{-1} \sqrt{\Theta H_{\text{END}} M_{\text{pl}}} \sim 5 \times 10^{-4} \sqrt{\Theta} M_{\text{pl}}, \quad (28)$$

where, once again, we have taken $H_{\text{END}} \sim 10^{-6} M_{\text{pl}}$.

2.2. Non-instantaneous decay

To better understand the gravitational reheating mechanism, we study in detail the decay process, using the dynamics described by the Boltzmann equations:

$$\frac{d\langle\rho(t)\rangle}{dt} + 3H\langle\rho(t)\rangle = -\Gamma\langle\rho(t)\rangle \quad \text{and} \quad \frac{d\langle\rho_r(t)\rangle}{dt} + 4H\langle\rho_r(t)\rangle = \Gamma\langle\rho(t)\rangle, \quad (29)$$

where $\langle\rho(t)\rangle$ continues denoting the energy density of the produced particles, $\langle\rho_r(t)\rangle$ the energy density of the decay products, i.e., the energy density of radiation and, the constant decay rate Γ represents the decay of heavy particles into light fermions. Specifically, $\Gamma \sim \frac{h^2 m_\chi}{8\pi}$, where h is a dimensionless constant [21].

We choose the following as the solution for the energy density of the massive particles:

$$\langle\rho(t)\rangle = \langle\rho_{\text{END}}\rangle \left(\frac{a_{\text{END}}}{a(t)}\right)^3 e^{-\Gamma(t-t_{\text{END}})}, \quad t \geq t_{\text{END}}, \quad (30)$$

Inserting this solution into the second equation of (29) and imposing that the decay begins at the end of inflation, we obtain:

$$\langle\rho_r(t)\rangle = \langle\rho_{\text{END}}\rangle \left(\frac{a_{\text{END}}}{a(t)}\right)^4 \int_{t_{\text{END}}}^t \frac{a(s)}{a_{\text{END}}} \Gamma e^{-\Gamma(s-t_{\text{END}})} ds. \quad (31)$$

We define the time at which decay ends, denoted as t_{dec} , as the time when $\Gamma(t_{\text{dec}} - t_{\text{END}}) \sim 1$, which gives $t_{\text{dec}} \sim t_{\text{END}} + \frac{1}{\Gamma}$, and we want to examine the evolution of the decay products when $t \gg t_{\text{dec}}$. Assuming that the background dominates, we will have $a(s) \cong a_{\text{END}} \left(\frac{s}{t_{\text{END}}}\right)^{\frac{n+1}{3n}}$, and taking into account that

$$\int_{t_{\text{END}}}^{\infty} (s/t_{\text{END}})^{\frac{n+1}{3n}} \Gamma e^{-\Gamma(s-t_{\text{END}})} ds \cong \bar{\Theta}^{-\frac{n+1}{6n}} \Gamma_E \left(\frac{4n+1}{3n}\right) \sim \bar{\Theta}^{-\frac{n+1}{6n}}, \quad (32)$$

where we have used the previous definition of $\bar{\Theta}$ and have denoted by Γ_E the Euler's Gamma function, we conclude that after the decay, i.e., for $t > t_{\text{dec}}$, we can safely make the approximation:

$$\langle\rho_r(t)\rangle \cong \langle\rho_{\text{END}}\rangle \bar{\Theta}^{-\frac{n+1}{6n}} \left(\frac{a_{\text{END}}}{a(t)}\right)^4. \quad (33)$$

On the other hand, recalling that the background evolves as

$$\rho_B(t) = 3H_{\text{END}}^2 M_{\text{pl}}^2 \left(\frac{a_{\text{END}}}{a(t)}\right)^{\frac{6n}{n+1}}, \quad (34)$$

and the universe becomes reheated when $\rho_B \sim \langle \rho_r \rangle$, we get:

$$\Theta = \bar{\Theta}^{\frac{n+1}{6n}} \left(\frac{a_{\text{END}}}{a_{\text{reh}}} \right)^{\frac{2(n-2)}{n+1}}, \quad (35)$$

and thus, the energy density of the decay products at the reheating time is:

$$\langle \rho_{r,\text{reh}} \rangle \cong 3H_{\text{END}}^2 M_{\text{pl}}^2 \bar{\Theta}^{-\frac{n+1}{2(n-2)}} \Theta^{\frac{3n}{n-2}}. \quad (36)$$

Therefore, from the Stefan-Boltzmann law

$$T_{\text{reh}}(n) = \left(\frac{30}{\pi^2 g_{\text{reh}}} \right)^{1/4} \langle \rho_{r,\text{reh}} \rangle^{1/4},$$

we obtain the following reheating temperature:

$$T_{\text{reh}}(n) \cong 5 \times 10^{-1} \left(\frac{\Theta^{3n}}{\bar{\Theta}^{\frac{n+1}{2}}} \right)^{\frac{1}{4(n-2)}} \sqrt{H_{\text{END}} M_{\text{pl}}}. \quad (37)$$

We also need to determine the range of values for the decay rate Γ . Since the decay occurs well after the end of inflation, we have $\Gamma \ll H_{\text{END}}$. On the other hand, we have assumed that by the end of the decay, the energy density of the background dominates, i.e., $\langle \rho_{r,\text{dec}} \rangle \ll \rho_{B,\text{dec}} \cong 3M_{\text{pl}}^2 \Gamma^2$. To get the bound, we use that the relation $\rho_{B,\text{dec}} \cong 3M_{\text{pl}}^2 \Gamma^2$, leads to:

$$\left(\frac{a_{\text{END}}}{a_{\text{dec}}} \right)^4 \cong \bar{\Theta}^{\frac{2(n+1)}{3n}}, \quad (38)$$

and thus, inserting it into the relation $\langle \rho_{r,\text{dec}} \rangle \ll 3M_{\text{pl}}^2 \Gamma^2$, we conclude that:

$$\Theta^{\frac{n}{n-1}} \ll \sqrt{\bar{\Theta}} \ll 1. \quad (39)$$

In addition, the combination of (39) with the bound of the reheating temperature $5 \times 10^{-22} M_{\text{pl}} \leq T_{\text{reh}}(n) \leq 5 \times 10^{-10} M_{\text{pl}}$ —what means that it remains within a range consistent with Big Bang Nucleosynthesis, which occurs around the scale of 1 MeV, and with an upper bound around 10^9 GeV in order to avoid issues related to the gravitino problem [22]—leads to four different cases. However, we have checked that the only viable one is when

$$\Theta^{\frac{n}{n-1}} \ll \sqrt{\bar{\Theta}} \leq 10^{\frac{84(n-2)}{n+1}} \left(\frac{H_{\text{END}}}{M_{\text{pl}}} \right)^{\frac{2(n-2)}{n+1}} \Theta^{\frac{3n}{n+1}}, \quad (40)$$

provided that:

$$10^{-\frac{42(n-1)}{n}} \left(\frac{M_{\text{pl}}}{H_{\text{END}}} \right)^{\frac{n-1}{n}} \ll \Theta \ll 10^{-\frac{28(n-2)}{n}} \left(\frac{M_{\text{pl}}}{H_{\text{END}}} \right)^{\frac{2(n-2)}{3n}}, \quad (41)$$

which for $H_{\text{END}} \cong 10^{-6} M_{\text{pl}}$ becomes:

$$10^{-\frac{36(n-1)}{n}} \ll \Theta \ll 10^{-\frac{24(n-2)}{n}}. \quad (42)$$

A final remark is in order: When the decay occurs well before to the onset of reheating one has $F_n(\Theta, \bar{\Theta}) \cong \bar{\Theta}^{\frac{n+1}{2n}}$. Therefore, in this situation, both reheating formulas (24) and (37) coincide.

3. Stokes phenomenon

As we have already explained, the key quantity to calculate is the energy density of the particles produced at the end of inflation [18, 19].

$$\langle \rho_{\text{END}} \rangle = \frac{m_\chi}{2\pi^2} \int_0^\infty k^2 |\beta_k|^2 dk, \quad (43)$$

where β_k is the β -Bogoliubov coefficient and, m_χ is the mass of the produced particles, which we will assume is small compared with the Hubble rate at the end of inflation H_{END} .

We focus on the class of potentials (1). Taking into account that when $m_\chi \ll H_{\text{END}}$ the main contribution of particle production comes from the pure Hubble expansion [23], we disregard the oscillating effect, and we approximate the scale factor, in the complex plane, by:

$$a = \begin{cases} -\frac{1}{H_{\text{END}}\eta} & \text{when } \text{Re}(\eta) < \eta_{\text{END}} \\ a_{\text{END}} \left[\frac{3n}{n+1} \left(1 - \frac{\eta}{\eta_{\text{END}}} \right) + \frac{\eta}{\eta_{\text{END}}} \right]^{\frac{n+1}{2n-1}} & \text{when } \text{Re}(\eta) \geq \eta_{\text{END}}, \end{cases} \quad (44)$$

with $a_{\text{END}} = -\frac{1}{H_{\text{END}}\eta_{\text{END}}}$, being $\eta_{\text{END}} < 0$. This model depicts a purely de Sitter phase during inflation with a sudden transition to a phase with a constant EoS parameter $w_{\text{eff}}(n) = \frac{n-1}{n+1}$. We have to recall that, for a smooth potential as (1), this phase transition is not so abrupt.

On the other hand, the complex WKB approximation tells us that the main contribution of the particle production comes from the turning points (Stokes phenomenon) —in our case points where $\omega_k^2(\eta) = k^2 + a^2(\eta)m_\chi^2 = 0$ — and at the end of inflation [6].

3.1. Turning points

The turning points corresponding to the de Sitter phase are $\eta_c = \pm i \frac{m_\chi}{H_{\text{END}} k}$, and since its real part is 0 they do not belong in the inflationary domain, and thus, they do not contribute to the particle production. The other turning points, corresponding to the other phase, are:

$$\eta_c = \frac{3n}{2n-1} \eta_{\text{END}} - (-1)^{\frac{2n-1}{2(n+1)}} \frac{n+1}{2n-1} \eta_{\text{END}} \left(\frac{k}{a_{\text{END}} m_\chi} \right)^{\frac{2n-1}{n+1}}. \quad (45)$$

Here we can see that for $n = 2$ the real part of the turning points is $2\eta_{\text{END}}$, which belongs in the inflationary domain, and thus, there is no contribution to the particle production. For the other values of n , when $k \gg a_{\text{END}} m_\chi$, we can see that there are turning points whose real part does not belong in the inflationary domain, and thus, they contribute to the particle production.

Next, we consider only the turning points with positive imaginary part and we take the straight line:

$$\eta = \frac{3n}{2n-1} \eta_{\text{END}} - (-1)^{\frac{2n-1}{2(n+1)}} \frac{n+1}{2n-1} \eta_{\text{END}} \left(\frac{k}{a_{\text{END}} m_\chi} \right)^{\frac{2n-1}{n+1}} \tau, \quad \text{with } 0 \leq \tau \leq 1. \quad (46)$$

Over this line $a(\tau) = i \frac{k}{m_\chi} \tau^{\frac{n+1}{2n-1}}$, and thus, $\omega_k^2(\tau) = k^2 (1 - \tau^{\frac{2(n+1)}{2n-1}})$. Therefore, based on the approach described in [24], we have:

$$|\beta_k|^2 \cong \exp \left(-4Im \int_{\frac{3n}{2n-1} \eta_{\text{END}}}^{\eta_c} \omega_k(z) dz \right), \quad (47)$$

where Im denotes the imaginary part and the integral is performed along the line (46). Then, we have:

$$|\beta_k|^2 \cong \exp \left(-c_n \left(\frac{k}{H_{\text{END}} a_{\text{EDN}}} \left(\frac{H_{\text{END}}}{m_\chi} \right)^{\frac{2n-1}{3n}} \right)^{\frac{3n}{n+1}} \right), \quad (48)$$

where the value of the dimensionless constant is:

$$\begin{aligned} c_n &= 4 \sin \left(\frac{\pi}{2(n+1)} \right) \frac{n+1}{2n-1} \int_0^1 \sqrt{1 - \tau^{\frac{2(n+1)}{2n-1}}} d\tau = \\ &= 2 \sin \left(\frac{\pi}{2(n+1)} \right) \int_0^1 t^{-\frac{3}{2n+2}} \sqrt{1-t} dt = \\ &= 2 \sin \left(\frac{\pi}{2(n+1)} \right) B \left(\frac{2n-1}{2n+2}, \frac{3}{2} \right), \end{aligned} \quad (49)$$

being B denotes the Euler's beta function, and from all the turning points with positive imaginary part we have chosen the one which the minimum value of the imaginary part (see for details the Section 52 of the Chapter VII in [24]).

Finally, its contribution to the energy density is:

$$C_n m_\chi H_{\text{END}}^3 \left(\frac{m_\chi}{H_{\text{END}}} \right)^{(2n-1)/n} \left(\frac{a_{\text{END}}}{a(t)} \right)^3, \quad (50)$$

where

$$\begin{aligned} C_n &\cong \frac{1}{2\pi^2} \int \left(\frac{m_\chi}{H_{\text{END}}} \right)^{\frac{n+1}{3n}} x^2 \exp\left(-c_n x^{\frac{3n}{n+1}}\right) dx \\ &\cong \frac{1}{2\pi^2} \int_0^\infty x^2 \exp\left(-c_n x^{\frac{3n}{n+1}}\right) dx. \end{aligned} \quad (51)$$

3.1.1. Case $n = 1$

As an example, we calculate:

$$c_1 = 4\sqrt{2} \int_0^1 \sqrt{1-\tau^4} d\tau = \sqrt{2} B\left(\frac{1}{4}, \frac{3}{2}\right) = \sqrt{2} \frac{\Gamma_E\left(\frac{1}{4}\right) \Gamma_E\left(\frac{3}{2}\right)}{\Gamma_E\left(\frac{7}{4}\right)}, \quad (52)$$

and since $\Gamma_E\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, and using the duplication formula [25]

$$\Gamma_E(z) \Gamma_E\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma_E(2z), \quad (53)$$

we find $\Gamma_E\left(\frac{7}{4}\right) = \frac{3\pi}{2\sqrt{2}} \Gamma_E^{-1}\left(\frac{1}{4}\right)$, and thus, we obtain the same result numerically obtained in [23] and analytically in the formula (B.4) of [15]:

$$c_1 = \frac{2}{3\sqrt{\pi}} \Gamma_E^2\left(\frac{1}{4}\right) \cong 4.9442, \quad (54)$$

where we have used that $\Gamma_E\left(\frac{1}{4}\right) \cong 3.6256$.

Once we have calculated c_1 , one has:

$$\begin{aligned} C_1 &\cong \frac{1}{2\pi^2} \int_0^\infty x^2 e^{-c_1 x^{3/2}} dx = \frac{1}{\pi^2 c_1^2} \int_0^\infty y^5 e^{-y^2} dy = \frac{1}{\pi^2 c_1^2} \\ &= \frac{9}{4\pi} \Gamma_E^{-4}\left(\frac{1}{4}\right) \cong 4 \times 10^{-3}. \end{aligned} \quad (55)$$

3.2. End of inflation

Concerning to the end of inflation, η_{END} , close to this point we make the quadratic approximation:

$$a(\eta) \cong a_{\text{END}} + a_{\text{END}}^2 H_{\text{END}} (\eta - \eta_{\text{END}}) + \frac{1}{2} a_{\text{END}}^3 H_{\text{END}}^2 (\eta - \eta_{\text{END}})^2, \quad (56)$$

and up to degree two, we have

$$a^2(\eta) \cong a_{\text{END}}^2 + 2a_{\text{END}}^3 H_{\text{END}} (\eta - \eta_{\text{END}}) + 2a_{\text{END}}^4 H_{\text{END}}^2 (\eta - \eta_{\text{END}})^2, \quad (57)$$

and the frequency can be approximated by

$$\omega_k^2(\eta) \cong k^2 + \frac{m_\chi^2 a_{\text{END}}^2}{2} + 2a_{\text{END}}^4 H_{\text{END}}^2 m_\chi^2 \left(\eta - \frac{3}{2} \eta_{\text{END}} \right)^2, \quad (58)$$

and defining

$$\tau \equiv \sqrt{\sqrt{2} a_{\text{END}}^2 H_{\text{END}} m_\chi} \left(\eta - \frac{3}{2} \eta_{\text{END}} \right), \quad (59)$$

the dynamical equation of the k -mode is

$$\frac{d^2 \chi_k}{d\tau^2} + (\kappa^2 + \tau^2) \chi_k = 0, \quad (60)$$

where we have introduced the notation $\kappa^2 = \frac{k^2 + a_{\text{END}}^2 m_\chi^2 / 2}{\sqrt{2} a_{\text{END}}^2 H_{\text{END}} m_\chi}$. Note that for this quadratic frequency the β -Bogoliubov coefficient is obtained using the well-known formula

$$|\beta_k|^2 = e^{-\pi \kappa^2} = \exp \left(-\pi \frac{k^2 + a_{\text{END}}^2 m_\chi^2 / 2}{\sqrt{2} a_{\text{END}}^2 H_{\text{END}} m_\chi} \right), \quad (61)$$

which can be derived as follows. First, recall that the positive frequency modes in the WKB approximation are

$$\phi_{k,+}(\tau) = \frac{1}{(\kappa^2 + \tau^2)^{1/4}} e^{-i \int \sqrt{\kappa^2 + \tau^2} d\tau}, \quad (62)$$

and for large values of $|\tau|$ ($|\tau| \gg \kappa$) one can make the approximations $(\kappa^2 + \tau^2)^{1/4} \cong |\tau|^{1/2}$ and $\sqrt{\kappa^2 + \tau^2} \cong |\tau| \left(1 + \frac{\kappa^2}{2\tau^2} \right)$, obtaining

$$\begin{aligned} \phi_{k,+}(\tau \ll -\kappa) &\cong |\tau|^{-1/2 + i\kappa^2/2} e^{i\tau^2/2}, \\ \phi_{k,+}(\tau \gg \kappa) &\cong |\tau|^{-1/2 - i\kappa^2/2} e^{-i\tau^2/2}, \end{aligned} \quad (63)$$

while for the negative frequency modes

$$\phi_{k,-}(\tau \gg \kappa) = \frac{1}{(\kappa^2 + \tau^2)^{1/4}} e^{i \int \sqrt{\kappa^2 + \tau^2} d\tau} \cong |\tau|^{-1/2 + i\kappa^2/2} e^{i\tau^2/2}. \quad (64)$$

On the other hand, the positive frequency modes evolve as

$$\phi_{k,+}(\tau \ll -\kappa) \longrightarrow \alpha_k \phi_{k,+}(\tau \gg \kappa) + \beta_k \phi_{k,-}(\tau \gg \kappa). \quad (65)$$

So, to calculate the Bogoliubov coefficients one can also use the WKB method in the complex plane integrating the frequency along the path $\gamma = \{z = |\tau|e^{i\alpha}, -\pi \leq \alpha \leq 0\}$, obtaining that for $\tau \gg \kappa$ the early time positive frequency modes evolve at late time as

$$e^{-\frac{\kappa^2}{2}\pi} |\tau|^{-1/2 + i\kappa^2/2} e^{i\tau^2/2}, \quad (66)$$

and comparing with (65) one gets

$$|\beta_k|^2 \cong e^{-\kappa^2\pi} \quad \text{and} \quad |\alpha_k|^2 = 1 + |\beta_k|^2 = 1 + e^{-\kappa^2\pi}. \quad (67)$$

Note that we can obtain the same result considering the turning point

$$\eta_c = \frac{3}{2}\eta_{\text{END}} + i \frac{\sqrt{k^2 + a_{\text{END}}^2} m_\chi^2/2}{\sqrt{2} a_{\text{END}}^2 H_{\text{END}} m_\chi}, \quad (68)$$

and calculating $4Im \int_{\frac{3}{2}\eta_{\text{END}}}^{\eta_c} \omega_k(z) dz$ over the line $\frac{3}{2}\eta_{\text{END}} + i \frac{\sqrt{k^2 + a_{\text{END}}^2} m_\chi^2/2}{\sqrt{2} a_{\text{END}}^2 H_{\text{END}} m_\chi} \tau$ with $0 \leq \tau \leq 1$, one gets $\pi\kappa^2$. Then, the complex WKB method also leads to the same result.

From this result, and considering the case $m_\chi \ll H_{\text{END}}$, we obtain that the contribution to the energy density of the produced particles, is:

$$\frac{1}{4\pi^3} m_\chi^2 H_{\text{END}}^2 \sqrt{\frac{m_\chi}{\sqrt{2} H_{\text{END}}}} \left(\frac{a_{\text{END}}}{a(t)} \right)^3. \quad (69)$$

3.3. Energy density of the produced particles at the end of inflation

Taking into account the contribution of the turning points and the end of inflation, we can conclude that, for $m_\chi \ll H_{\text{END}}$, the energy density of the scalar produced particles at the end of inflation is:

$$\langle \rho_{\text{END}} \rangle \cong \begin{cases} \frac{9}{4\pi} \Gamma_E^{-4} \left(\frac{1}{4} \right) m_\chi^2 H_{\text{END}}^2 & \text{for } n = 1 \\ \frac{1}{4\pi^3} m_\chi^2 H_{\text{END}}^2 \sqrt{\frac{m_\chi}{\sqrt{2} H_{\text{END}}}} & \text{for } n \neq 1, \end{cases} \quad (70)$$

where, we have used that, for $n \neq 1$, the main contribution to the particle production is at the end of inflation, because, since we are dealing with particles with mass lower than the Hubble rate at the end of inflation, the contribution $C_n m_\chi H_{\text{END}}^3 \left(\frac{m_\chi}{H_{\text{END}}}\right)^{(2n-1)/n}$ decreases as n increases, so the dominant term is for $n = 2$ which is of the same order as the contribution produced close to η_{END} .

3.3.1. Non-instantaneous decay

Considering the potential (1), and taking into account that inflation ends when the main slow roll parameter is 1, a simple calculation leads to $H_{\text{END}} \cong 5 \left(1 - \frac{2\sqrt{3n-3}}{4n^2-3}\right)^n 10^{-6} M_{\text{pl}}$, which for all value of n is close to $2 \times 10^{-6} M_{\text{pl}}$, and from the formula of the energy density of the produced particles at the end of inflation, i.e, $\langle \rho_{\text{END}} \rangle \cong \frac{1}{4\pi^3} m_\chi^2 H_{\text{END}}^2 \sqrt{\frac{m_\chi}{\sqrt{2}H_{\text{END}}}}$, for $m_\chi \ll H_{\text{END}}$, we have:

$$\Theta = \frac{1}{12\pi^3} \left(\frac{m_\chi}{M_{\text{pl}}}\right)^2 \sqrt{\frac{m_\chi}{\sqrt{2}H_{\text{END}}}} \cong 6 \times 10^{-1} \left(\frac{m_\chi}{M_{\text{pl}}}\right)^{5/2}. \quad (71)$$

Therefore, the reheating temperature, as a function of the mass of the produced particles and the decay rate, becomes:

$$T_{\text{reh}}(n) \cong 5 \times 10^{-1} \left(\frac{H_{\text{END}}}{\Gamma}\right)^{\frac{n+1}{4(n-2)}} \left(\frac{m_\chi}{M_{\text{pl}}}\right)^{\frac{15n}{8(n-2)}} M_{\text{pl}}, \quad (72)$$

with the constraints:

$$3 \times 10^{-\frac{14(n-1)}{n}} \ll \frac{m_\chi}{M_{\text{pl}}} \ll 2 \times 10^{-\frac{10(n-1)}{n}} \quad (73)$$

and the one given by (40)

$$\left(\frac{m_\chi}{M_{\text{pl}}}\right)^{\frac{5n}{2(n-1)}} \ll \frac{\Gamma}{H_{\text{END}}} \leq \frac{2}{5} \times 10^{\frac{72(n-2)}{n+1}} \left(\frac{m_\chi}{M_{\text{pl}}}\right)^{\frac{15n}{2(n+1)}}. \quad (74)$$

On the contrary, assuming that the decay is nearly instantaneous and close to the radition epoch, we find the following expression of the maximum reheating temperature as a function of the mass m_χ :

$$T_{\text{reh}}^{\text{max}}(n) \cong 10^{-3} \left(\frac{m_\chi}{M_{\text{pl}}}\right)^{\frac{5n}{4(n-1)}} M_{\text{pl}} \quad (75)$$

with

$$2 \times 10^{-\frac{15(n-1)}{n}} \leq \frac{m_X}{M_{\text{pl}}} \ll 2 \times 10^{-6}, \quad (76)$$

coming to the imposition that the reheating temperature is in the range of 1 MeV to 10^9 GeV.

4. Gravitational dark matter production

In this section, we study the gravitational production of dark matter within the context of gravitational reheating. Specifically, we consider two quantum scalar fields, X and Y . The X -field is responsible for the gravitational production of X -particles with mass m_X , which will decay into Standard Model particles to reheat the universe. The Y -field, on the other hand, is responsible for the production of Y -particles with mass m_Y , which will account for present-day dark matter.

Assuming that the decay of the X -particles occurs during the inflaton domination. At the onset of radiation the energy density of the X -particles is (36):

$$\langle \rho_{X,\text{reh}} \rangle \cong 3H_{\text{END}}^2 M_{\text{pl}}^2 \bar{\Theta}_X^{-\frac{n+1}{2(n-2)}} \Theta_X^{\frac{3n}{n-2}}, \quad (77)$$

where Θ_X and $\bar{\Theta}_X$ are the parameters corresponding to the X -particles.

At the matter-radiation equality, which we will denote by "eq", we will have:

$$\frac{a_{\text{reh}}}{a_{\text{eq}}} = \frac{\langle \rho_{Y,\text{reh}} \rangle}{\langle \rho_{X,\text{reh}} \rangle} \implies \langle \rho_{Y,\text{eq}} \rangle = \frac{\langle \rho_{Y,\text{reh}} \rangle^4}{\langle \rho_{X,\text{reh}} \rangle^3}. \quad (78)$$

Next, we have to take into account that

$$\langle \rho_{Y,\text{reh}} \rangle = \langle \rho_{Y,\text{END}} \rangle \left(\frac{a_{\text{END}}}{a_{\text{reh}}} \right)^3 = 3H_{\text{END}}^2 M_{\text{pl}}^2 \bar{\Theta}_X^{-\frac{(n+1)^2}{4n(n-2)}} \Theta_X^{\frac{3(n+1)}{2(n-2)}} \Theta_Y, \quad (79)$$

where we have used the relation (35):

$$\Theta_X = \bar{\Theta}_X^{\frac{n+1}{6n}} \left(\frac{a_{\text{END}}}{a_{\text{reh}}} \right)^{\frac{2(n-2)}{n+1}}. \quad (80)$$

Therefore:

$$\langle \rho_{Y,\text{eq}} \rangle = 3H_{\text{END}}^2 M_{\text{pl}}^2 \bar{\Theta}_X^{\frac{n+1}{2n}} \Theta_X^{-3} \Theta_Y^4. \quad (81)$$

On the other hand, considering the central values of the red shift at the matter-radiation equality $z_{\text{eq}} = 3365$, the present value of the ratio of the matter energy density to the critical one $\Omega_{\text{m},0} = 0.308$, and $H_0 = 67.81$ Km/sec/Mpc, one can deduce that the present value of the matter energy density is $\rho_{\text{m},0} = 3H_0^2 M_{\text{pl}}^2 \Omega_{\text{m},0} \cong 3 \times 10^{-121} M_{\text{pl}}^4$, and at matter radiation equality one will have $\rho_{\text{m,eq}} = \rho_{\text{m},0}(1 + z_{\text{eq}})^3 \cong 10^{-110} M_{\text{pl}}^4$. Since practically all the matter has a not baryonic origin, one can conclude that $\langle \rho_{Y,\text{eq}} \rangle \cong \rho_{\text{m,eq}}$, and thus, we have the relation between the decay rate of the X -particles and the heating efficiencies.

Then,

$$\sqrt{\bar{\Theta}} \cong \left(\frac{\Theta_X^3}{12\Theta_Y^4} \right)^{\frac{n}{n+1}} 10^{-\frac{98n}{n+1}}, \quad (82)$$

where we have used that $H_{\text{END}} \cong 2 \times 10^{-6} M_{\text{pl}}$, and from the constraint (74), we get:

$$10^{99} \left(\frac{m_X}{M_{\text{pl}}} \right)^{\frac{5(n+1)}{2(n-1)}} \ll \frac{\Theta_X^3}{\Theta_Y^4} \leq 5 \times 10^{\frac{2(85n-72)}{n}} \left(\frac{m_X}{M_{\text{pl}}} \right)^{\frac{15}{2}} \quad (83)$$

Next, from (71), we have $\Theta_A \cong 6 \times 10^{-1} \left(\frac{m_A}{M_{\text{pl}}} \right)^{5/2}$, with $A = X, Y$, what leads to the following relation between the masses:

$$10^{-17} \times 10^{\frac{72}{5n}} \leq \frac{m_Y}{M_{\text{pl}}} \ll 10^{-10} \left(\frac{m_X}{M_{\text{pl}}} \right)^{\frac{n-2}{2(n-1)}}. \quad (84)$$

4.1. Dark matter production when the reheating temperature is maximum

As we have already shown, the reheating temperature attains its maximum value when the decay is nearly instantaneous and it occurs close to the end of the inflaton's domination. In this situation, the energy density of the X -field at the onset of radiation epoch is $\langle \rho_{X,\text{reh}} \rangle = 3H_{\text{END}}^2 M_{\text{pl}}^2 \Theta_X^{\frac{2n}{n-1}}$ and the one of the Y -field is $\langle \rho_{Y,\text{reh}} \rangle = 3H_{\text{END}}^2 M_{\text{pl}}^2 \Theta_X^{\frac{n+1}{n-1}} \Theta_Y$, what leads to

$$\langle \rho_{Y,\text{eq}} \rangle = 3H_{\text{END}}^2 M_{\text{pl}}^2 \Theta_X^{\frac{-2(n-2)}{n-1}} \Theta_Y^4. \quad (85)$$

And thus,

$$3 \frac{H_{\text{END}}^2}{M_{\text{pl}}^2} \Theta_X^{\frac{-2(n-2)}{n-1}} \Theta_Y^4 \cong 10^{-110} \implies 12 \Theta_X^{\frac{-2(n-2)}{n-1}} \Theta_Y^4 \cong 10^{-98} \quad (86)$$

What leads to the following link between both masses

$$\frac{m_Y}{M_{\text{pl}}} \cong 10^{-10} \left(\frac{m_X}{M_{\text{pl}}} \right)^{\frac{n-2}{2(n-1)}}. \quad (87)$$

5. Conclusions

In this study, we investigated the reheating temperature for inflationary potentials where gravitational reheating—enabled by the production of massive particles conformally coupled to gravity—can take place effectively. We found that for gravitational reheating to work during inflaton oscillations, the Universe must have an effective equation of state (EoS) parameter greater than $1/3$. This condition ensures the inflaton’s energy density decreases fast enough to avoid re-domination after the initial reheating phase, a crucial requirement for post-inflationary evolution to match the standard Λ CDM model.

We derived formulas for the reheating temperature in two cases: when the decay of heavy particles is nearly instantaneous and when it is slower. We also found an expression for the maximum reheating temperature, which occurs when decay starts as soon as the produced particles begin to dominate the energy density. Using the Stokes phenomenon, we calculated the energy density of produced particles at the end of inflation as a function of their mass, allowing us to set limits on particle masses that yield a viable reheating temperature.

Additionally, we examined the gravitational production of dark matter, deriving relationships between the mass of particles responsible for reheating and the mass of dark matter particles. These findings naturally link the reheating process to the origin of dark matter, providing constraints that enhance our understanding of both early-universe dynamics and the genesis of dark matter within a gravitational reheating framework.

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