Deformed Commutation Relations in Quantum Cosmology

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Abstract

We present a brief overview of the formalism of Deformed Commutation Relations, and of how it can be used to implement cut-off effects coming from Quantum Gravity theories. After introducing the forms that are relevant for Quantum Gravity and presenting their relevant properties, we implement them on various cosmological models in their semiclassical limit. In particular, we show how their implementation to the isotropic universe is sometimes able to remove singularities through a Quantum Bounce similar to Loop Quantum Cosmology. When implemented on the anisotropic Bianchi I model, the same Quantum Bounce can be interpreted as a new kind of Kasner transition. Finally, we show how their implementation on the anisotropic sector of the Bianchi IX model is able to remove the chaotic behaviour that is present at the classical level.

1. Introduction

General Relativity predicts its own breakdown in the form of spacetime singularities, but quantum effects are expected to become relevant in the high-energy regimes before singularities are reached. There have been many attempts to unify General Relativity (GR) and Quantum Mechanics (QM) in a theory of Quantum Gravity (QG); the most successful until now have been Loop Quantum Gravity (LQG) [1] and String Theories (STs) [2]. However, they still have some problems, besides being mathematically very convoluted and not easily applicable to more specific settings such as Black Holes or Cosmological Models. As a consequence, the formalism of Deformed Commutation Relations (DCRs) was developed.

DCRs are modifications of the standard Heisenberg commutator with higher-order terms that are supposed to become relevant only in the appropriate regime. Their main objective is to easily implement on any Hamiltonian and Quantum system some effects expected in more fundamental QG

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theories, such as for example cut-off effects in the form of a minimum length or a maximum energy [3]. In this respect, the most useful deformations are functions of the momentum. Some specific functions can be used to reproduce the same dynamics of other more fundamental theories, such as Loop Quantum Cosmology (LQC) or Brane Cosmology (a cosmological sector of String Theories) [4]. Furthermore, their straightforward classical limit, in which Deformed Commutation Relations get downgraded to Deformed Poisson Brackets, makes them extremely versatile.

Here we will implement four different Algebras to a few different cosmological models. In particular, we will show the isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model, on which the most accepted description of our observable Universe is based, then the anisotropic Bianchi I, Bianchi II and the chaotic Bianchi IX models. The implementations will be somewhat different, in that for FLRW and Bianchi I and II we will deform the volume variable (i.e. the cubed scale factor), while for Bianchi IX we will deform only the anisotropic sector. When deforming the volume, we will focus on the possibility of the removal of singularities, and in particular in Bianchi I and II we will show how the Quantum Bounce can be interpreted as a different type of Kasner transition, complete with its own map for the Kasner indices. On the other hand, when deforming the anisotropies in Bianchi IX, we will focus on the fate of chaos and on whether the quantum corrections will keep it or remove it. All these results are obtained within the classical limit of the deformation, and where possible we will reference the full quantum dynamics, which however is still far from complete.

The manuscript is organized as follows. In Section 2. we will introduce the Deformed Commutation Relations. In Section 3. we will implement them on the various cosmological models mentioned above. Section 4. concludes the paper with a brief summary and outlook.

We use natural units $\hbar = c = 8\pi G = 1$.

2. Deformed Commutation Relations

We will start by introducing the general properties of Deformed Commutation Relations, and will then focus on some specific forms. As mentioned above, the aim is to reproduce properties and effects that are expected in Quantum Gravity theories, and in this respect the most commonly used deformations are functions of the momentum:

$$[\hat{q}, \hat{p}] = i\, f(\hat{p}), \tag{1}$$

where q and p represent two generic conjugate observables. Depending on the function f, it is possible to obtain various properties. Of course the commutators will reduce to the standard Heisenberg one in the appropriate limit.

The first consequence of the DCRs is a modified action for the fundamental operators. Given a specific function f, there are two main different operatorial representations (both in the momentum polarization):

$$1)\quad \hat{q}\,\psi(p) = i\,f(p)\,\frac{\mathrm{d}\psi}{\mathrm{d}p}, \qquad \hat{p}\,\psi(p) = p\,\psi(p), \tag{2a}$$

$$2)\quad \hat{q}\,\psi(p) = i\,\frac{\mathrm{d}\psi}{\mathrm{d}p}, \qquad \hat{p}\,\psi(p) = g(p)\,\psi(p). \tag{2b}$$

The function g of representation 2 is found as $g^{-1} = \int dp/f(p)$, while representation 1 also needs a modified measure $dp/f(p)$ in the integrals for expectation values in order to maintain the symmetry of the operators. In one-dimensional (i.e. isotropic) cosmology, the two representation seem to yield equivalent dynamics [4]. However it is not yet clear if they are actually equivalent in general.

An interesting property of the DCRs comes from their extension to higher space dimensions. While the commutators between q and p are simply generalized as

$$[\hat{q}_i, \hat{p}_j] = i\,\delta_{ij}\,f(\hat{p}_{\mathrm{tot}}), \qquad p_{\mathrm{tot}}^2 = \sum_i p_i^2, \tag{3}$$

the imposition of the Jacobi identities naturally yields a non commutativity between the space variables [5]:

$$[\hat{q}_i, \hat{q}_j] = i\,\frac{f'(\hat{p}_{\mathrm{tot}})\,\hat{J}_{ij}}{\hat{p}_{\mathrm{tot}}}, \qquad J_{ij} = q_i\,p_j - q_j\,p_i, \tag{4}$$

where J_{ij} is the angular momentum. However, there are some problems with operator ordering that undermine the validity of some properties usually valid in standard Quantum Mechanics. Therefore in this work we will restrict ourselves to the classical limit, and add some references to the full quantum dynamics where possible.

Indeed the classical limit is one of the most powerful features of the DCRs. It is sufficient to downgrade the Deformed Commutators to Deformed Poisson brackets:

$$\{q_i, p_j\} = \delta_{ij}\,f(p_{\mathrm{tot}}), \qquad \{q_i, q_j\} = \frac{f'(p_{\mathrm{tot}})\,J_{ij}}{p_{\mathrm{tot}}}. \tag{5}$$

The reduction to one single space dimension is trivial. This way it is straightforward to implement QG effects on any Hamiltonian system, and to easily make a comparison with classical General Relativity in its Hamiltonian formulation.

Table 1 shows the four forms that will be implemented on cosmological models below. The KMM Algebra is the original and most famous one, named after the authors Kempf, Mangano and Mann that first developed the so-called Generalized Uncertainty Principle representation (GUP) [3];

Algebra	f	QG effect
KMM	$1+\mu^2p^2$	$\Delta q \geq \Delta q_{\min} = \mu$
Loop	$\sqrt{1-\mu^2p^1}$	$\langle\hat{p}\rangle \leq \frac{1}{\mu}$
Brane	$\sqrt{1+\mu^2p^2}$	$\Delta q_{\min}$ (sometimes)
LUP	$1-\mu^2p^2$	$\langle\hat{p}\rangle \leq \frac{1}{\mu}$ (unclear)

Table 1: The four Algebras used in this paper. Explanations are in the text.

it implies an absolute minimal uncertainty on position $\Delta q_{\min}$. The Loop Algebra introduces a maximum momentum, while the Brane Algebra also has a minimal uncertainty but only under certain conditions; the names are due to their ability to reproduce effective Loop Quantum Cosmology and Brane Cosmology respectively [4]. LUP stands for Loop Uncertainty Principle, and it is a hybrid between the Loop and the KMM Algebras; there is a peculiar value for the momentum, but it is unclear whether it is an actual bound. The concepts of a minimal length and of an energy cut-off (as well as spatial Non-Commutativity) are predicted and expected in various theories of Quantum Gravity, and these four DCRs can implement them in a natural way.

3. Deformed Cosmology

In this Section we will implement the DCRs to various cosmological models, briefly showing the most important results compared to the corresponding classical dynamics.

3.1. Isotropic FLRW Model

Let us start from the isotropic FLRW model. Instead of the standard scale factor a, for reasons that will be explained later it is useful to change to the volume variable $v = a^3$. The Hamiltonian then is

$$N\,\mathcal{H} = N\left(-\frac{3}{4}\,p_v^2\,v + \rho\,v\right) = 0, \tag{6}$$

where p_v is the momentum conjugate to the volume, ρ is a generic energy density, and $N = N(t)$ is the Lapse function parametrizing the freedom to choose a time variable. Note how the whole Hamiltonian is constrained to zero; this is a symptom of the nature of time in General Relativity, and on a quantum level this leads to the Problem of Time. To work around this, it is often useful to introduce matter in the form of a free massless scalar field ϕ, with $\rho_\phi = p_\phi^2/2v^2$ (p_ϕ being the momentum conjugate to the field). Then it is possible to change time variable to the field itself: setting $N = v/p_\phi$ yields $\dot\phi = 1$, and the system can be evolved with respect to ϕ.

Starting from the classical case with standard Poisson brackets $\{v, p_v\} = 1$, using the constraint in the equations of motion yields the Friedmann equation, which governs the dynamics by linking the expansion rate to the matter content. In synchronous time t, it takes the form

$$H^2 = \left(\frac{1}{3}\frac{\dot v}{v}\right)^2 = \frac{\rho_\phi}{3}. \tag{7}$$

Since the energy density can be expressed as function of the volume (for a free massless field, p_ϕ is a constant), this becomes a differential equation that can be easily solved for $v(t)$:

$$v(t) = \pm\sqrt{\frac{3}{2}}\, p_\phi\,(t - t_0), \tag{8}$$

where the different signs indicate an expending or contracting Universe. It is clear how $t = t_0$ corresponds to singularities where $v = 0$ and $\rho_\phi \to \infty$. Using instead the scalar field time, the Friedmann equation greatly simplify, yielding an exponential solution:

$$\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = \frac{3}{2}, \qquad v(\phi) \propto \exp\left(\pm\sqrt{\frac{3}{2}}\,(\phi - \phi_0)\right). \tag{9}$$

The singularities have been moved to $\phi \to \pm\infty$.

Let us now consider the Deformed Poisson brackets $\{v, p_v\} = f(p_v)$. It is clear that they will change the equations of motion and, given the constraint linking p_v to the scalar field energy density, the Friedmann equations. Here are the four modified Friedmann equations and the solution for the volume in scalar field time for the four cases:

$$H^2_{\mathrm{KMM}} = \frac{\rho_\phi}{3}\left(1 + \frac{\rho_\phi}{\rho_\mu}\right)^2, \qquad v_{\mathrm{KMM}}(\phi) \propto \sqrt{e^{\pm 2\sqrt{\frac{3}{2}}\,(\phi-\phi_0)} - 1}\,, \tag{10a}$$

$$H^2_{\mathrm{Loop}} = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{\rho_\mu}\right), \qquad v_{\mathrm{Loop}}(\phi) \propto \cosh\left(\sqrt{\frac{3}{2}}\,(\phi - \phi_0)\right), \tag{10b}$$

$$H^2_{\mathrm{Brane}} = \frac{\rho_\phi}{3}\left(1 + \frac{\rho_\phi}{\rho_\mu}\right), \qquad v_{\mathrm{Brane}}(\phi) \propto \sinh\left(\sqrt{\frac{3}{2}}\,(\phi - \phi_0)\right),, \tag{10c}$$

$$H^2_{\mathrm{LUP}} = \frac{\rho_\phi}{3}\left(1 - \frac{\rho_\phi}{\rho_\mu}\right)^2, \qquad v_{\mathrm{LUP}}(\phi) \propto \sqrt{e^{\pm 2\sqrt{\frac{3}{2}}\,(\phi-\phi_0)} + 1}\,. \tag{10d}$$

In all correction factors, the constant critical density $\rho_\mu = 3/4\mu^2$ appears. This is true only when using the volume variable; with any other variable,

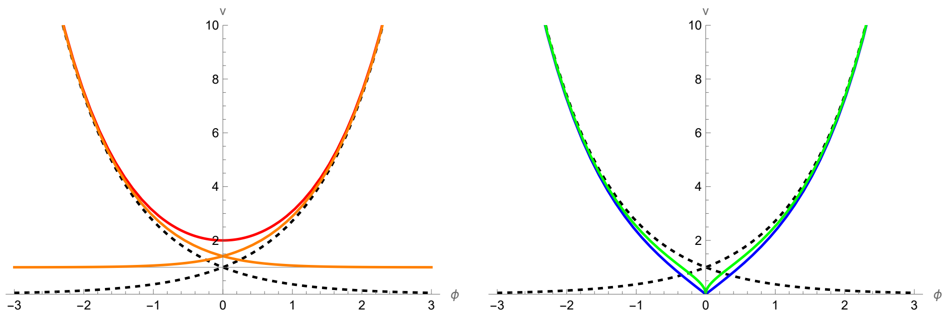

Figure 1: The evolution of the volume $v(\phi)$ in the four cases, compared with the classical exponential case (dashed black lines). Left: Loop (red) and LUP (orange). Right: KMM (green) and Brane (Blue).

the critical density would depend on the scale factor [6]. Furthermore, in the limit $\mu \to 0$, $\rho\mu \to \infty$ and the standard Friedmann equation is recovered. Regarding the dynamics, the corrections with the minus sign clearly introduce a critical point when $\rho_\phi = \rho_\mu$. Indeed, as evident from the solutions and from the left panel of Figure 1, singularities are removed; in particular, the Loop Algebra yields the same modified Friedmann equation of effective Loop Quantum Cosmology and the same Bouncing dynamics [7], while the LUP Algebra introduces an asymptote and reproduces the so-called Emergent Universe [8]. On the other hand, the two Algebras with the plus sign are not able to remove the singularities, but only to change how it is approached; in particular, the Brane Algebra reproduces the same Friedmann equation and dynamics of Brane Cosmology [4], with the difference that the role of the critical density ρ_μ is instead played by the Brane tension. The same results can be obtained on the full quantum level, where the constructed wave-packets follow the classical trajectories. For more details about this whole section, see [4].

It is interesting to note how an energy cut-off in the form of a maximum momentum is able to remove singularities, but a length cut-off in the form of a minimal uncertainty is not. This is probably due to the fact that the momentum cut-off is directly related to a cut-off on the energy density which is a physical observable, while the minimal uncertainty is implemented on the volume variable which is a metric quantity. Furthermore, as mentioned above, the resulting dynamics strongly depends on the variable chosen to be deformed: by using a different function of the scale factor instead of the volume, the critical density is not a constant any more, the removal of singularities is possible only for some types of matter, or sometimes it doesn't even happen at all. Some equivalence between different variables can be recovered by performing a transformation *after* the deformation, but then the new Poisson brackets will depend also on the variable and not just on the momentum, which can bring many complications on a quantum level

[9]. These issues are of a delicate and non-trivial nature, and need to be addressed on a full quantum level.

3.2. Anisotropic Bianchi I and II Models

The Bianchi models are nine classes of anisotropic (but still homogeneous) universes. They are therefore described by three different scale factors a_1, a_2 and a_3, one for each spatial direction. However, they are often studied in Misner or Misner-like variables, which include an isotropic bulk variable (the proper Misner description uses the logarithmic α, but we will often use the volume v instead) and the anisotropies β_+ and β_- which parametrise the (logarithmic) difference in evolution between the different directions [10]. The advantage of the Misner variables is that the kinetic term in the Hamiltonian becomes diagonal; in the case of the volume variable, we have

$$N\mathcal{H} = N\left(-\frac{3}{4}\,p_v^2 v + \frac{p_+^2 + p_-^2}{12v} + v\,U\right) = 0, \tag{11}$$

where $p_\pm$ are the momenta conjugate to the anisotropies, while U is a potential depending on the specific Bianchi model. We will start with the classical Bianchi I and II models, and then implement the DCRs.

The simplest model is Bianchi I, where the potential is zero; this implies that the anisotropy momenta $p_\pm$ are constants of motion. Using synchronous time t with $N = 1$, it is easy to see that the volume evolves linearly with time, while the anisotropies behave logarithmically. However it is better to use the harmonic time variable τ, defined as $\Box\tau = 0$ which corresponds to setting $N = v$ (note that here we are dealing with vacuum models, and therefore we do not have a scalar field to use as time). In this way, the volume becomes an exponential, while the anisotropies become linear:

$$v(\tau) \propto e^{v_1\tau}, \qquad \beta_\pm(\tau) = \overline{\beta_\pm} + \frac{p_\pm}{6}\,\tau. \tag{12}$$

With harmonic time τ, the dynamics of the Bianchi I model is that of a free point particle moving at constant speed in the (β_+, β_-) plane. If we go back to the three scale factors, the dynamics is the described as

$$a_i \propto t^{k_i}, \qquad \sum_i k_i = \sum_i k_i^2 = 1, \tag{13}$$

where the k_i are called Kasner indices and obey the above relations [11]. Note that, in order for them to be valid, one of the indices has to be negative; the Bianchi I dynamics is then an expanding universe where two space directions expand and the third one contracts, while the total volume grows linearly.

The classical Bianchi II model is the same as Bianchi I, but with a potential in the form of an exponentially steep wall parallel to the β_-

direction:

$$U_{\mathrm{BII}} = \frac{e^{-8\beta_+}}{4v^{\frac{2}{3}}}. \tag{14}$$

First of all, note how the presence of β_+ implies that the momentum p_+ is not constant any more, while p_- still is. Given the very steep exponential, the wall is negligible in the region $\beta_+ > 0$. Therefore the dynamics will be the same as Bianchi I, until the particle universe gets close to the potential wall; it gets then reflected, changing its direction into another (different) Bianchi I-like linear motion. The Bianchi II dynamics is therefore made by two different Bianchi I evolutions, smoothly linked by this reflection off of the potential wall. Using the integrals of motion $\Omega_1 = p_-$ and $\Omega_2 = 6p_v v + p_+$, it is possible to derive the following reflection law:

$$\sin(\theta_i + \theta_f) = 2(\sin\theta_i - \sin\theta_f), \tag{15}$$

where i and f stand for initial and final, and the angles are measured with respect to the perpendicular to the wall. In terms of the scale factors, the reflection has the effect of changing the Kasner indices according to the following map:

$$k_1' = \frac{k_1 + 2k_3}{1 + 2k_3}, \quad k_2' = \frac{k_2 + 2k_3}{1 + 2k_3}, \quad k_3' = -\frac{k_3}{1 + 2k_3}, \tag{16}$$

where $k_3 < 0$ implies $k3' > 0$ and $k_2' < 0$, meaning that the contraction has shifted from one direction to a different one. The smooth transition is even more evident when using harmonic time τ, thanks to which it is possible to obtain analytical solutions:

$$v(\tau) = e^{\overline{p_+}\,\tau} \sqrt{\frac{\cosh\Big(v_k(\tau - \tau_k)\Big)}{v_k}}\ , \tag{17}$$

$$\beta_+(\tau) = \frac{\overline{p_+}}{6}\tau + \frac{1}{3}\log\left(\frac{\cosh\Big(v_k(\tau - \tau_k)\Big)}{v_k}\right), \tag{18}$$

where τ_k is the time of the reflection, $\overline{p_+} = p_+(\tau = \tau_k)$ is the initial value for the not-constant p_+, and $v_k^2 = \overline{p_+}^2 - \overline{p_-}^2/3$. From the evolution of the volume, it is easy to see that the asymptotic behaviours for $\tau \ll \tau_k$ and $\tau \gg \tau_k$ are different exponentials, corresponding to two different Bianchi I models.

Let us now deform the Bianchi I model. In particular, we deform the volume variable with the Loop Algebra (it is possible to separately deform the anisotropic sector at the same time, but in Bianchi I this does not really affect the dynamics since the variables $\beta_\pm$ do not appear in the Hamiltonian). Then the Friedmann equation for the volume takes the same

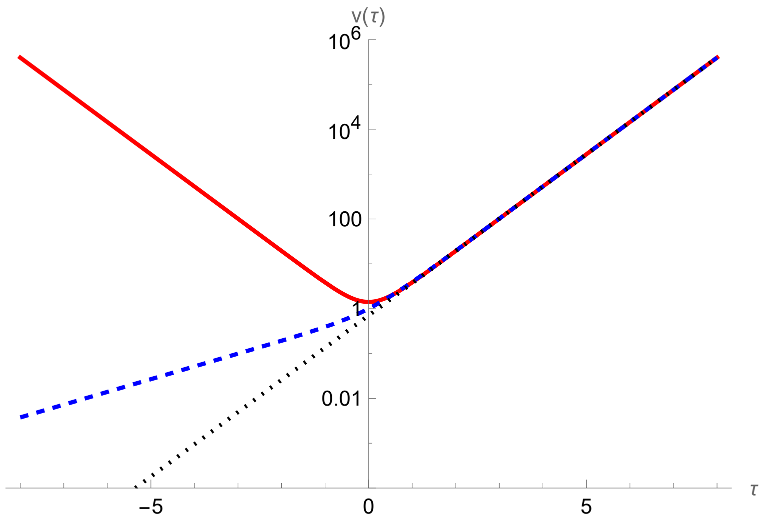

Figure 2: Evolution of the volume v as function of harmonic time τ for the Bouncing Bianchi I model (red continuous line), compared with the classical Bianchi II (blue dashed) and Bianchi I (black dotted) models.

form (10b) of the isotropic case, and the volume singularity is replaced by a Big Bounce. However, an interesting property emerges when using harmonic time τ. In particular, the solution for the volume is

$$v(\tau) = v_B \cosh(v_1 \tau), \tag{19}$$

which looks similar to the solution for Bianchi II. Indeed, the Bouncing Bianchi I model could be seen as a smooth transition between two classical Bianchi I models, one contracting and one expanding; this is nothing but a (different kind of) Kasner transition. Furthermore, by going back to synchronous time t and relating the volume to the directional scale factors, it is possible to obtain a new map between the Kasner indices before and after the Bounce:

$$k_i' = \frac{2}{3} - k_i. \tag{20}$$

Figure 2 compares a Bouncing Bianchi I model, a classical Bianchi II and a classical Bianchi I, showing how the first two are just similar kinds of smooth transitions between classical Bianchi I models. This interpretation and the above map are consistent with what was found in anisotropic LQC [12], meaning that this might be a general property of this kind of quantum Bounce. For more information on this study, see [13].

3.3. Bianchi IX Mixmaster Model

The Bianchi IX model is the most general homogeneous cosmological model. Its relevance lies in the fact that its generality makes it the perfect arena to test quantum corrections, and it can be used as the starting point for the general inhomogeneous cosmological solution. When considering Misner variables, its potential U in the (β_+, β_-) plane is constituted by three exponentially steep walls with the symmetry of an equilateral triangle. Therefore all the considerations made for the Bianchi II reflection are still valid, including the reflection law, but the point universe will keep being infinitely reflected off of the different walls and acquires a chaotic dynamics, earning it the nickname Mixmaster universe [10].

Here we are choosing to deform the anisotropic sector, meaning that we will have

$$\{\beta_\pm, p_\pm\} = \delta_\pm f(p_{\text{tot}}), \quad \{\beta_+, \beta_-\} = \frac{f'(p_{\text{tot}})(\beta_+ p_- - \beta_- p_+)}{p_{\text{tot}}}, \tag{21}$$

where $p_{\text{tot}}^2 = p_+^2 + p_-^2$ and f is one of the DCRs introduced in Section 2.. The isotropic sector identified by the volume variable will be left alone; as mentioned above, it is possible to deform both sectors at the same time, but in that case it is not possible to obtain a closed form for the constants of motion $\Omega_{1,2}$ mentioned above, and only a numerical solution is possible [13].

Since the isotropic sector is left untouched, we will be using pure Misner variables, meaning that we will have $\alpha = \ln(v)/3$ instead of v; furthermore, we will use this logarithmic variable as internal time, by setting $N = -p_\alpha/6e^{3\alpha}$ and performing a reduction of the dynamics. The (deformed) integrals of motion in the Bianchi II approximation then are

$$\Omega_1 = p_-, \qquad \Omega_2 = 2p_\alpha + \int \frac{dp_+}{f(p_{\text{tot}})}. \tag{22}$$

Note how in the standard case i.e. for $f = 1$ the classical integral $\Omega_2 = 2p_\alpha + p_+$ is recovered. The reflection maps will then be modified; in particular, each of the different Algebras will have the equation $p_\alpha^i \sin\theta_i =$

$p_\alpha^f \sin\theta_f$, supported by another equation depending on the function f:

$$\begin{aligned}\text{Brane:}\quad & p_\alpha^i + \frac{1}{2\mu}\text{arctanh}\left(\frac{\mu\, p_\alpha^i \cos\theta_i}{\sqrt{1+\mu^2 (p_\alpha^i)^2}}\right)\\ &= p_\alpha^f - \frac{1}{2\mu}\text{arctanh}\left(\frac{\mu\, p_\alpha^f \cos\theta_f}{\sqrt{1+\mu^2 (p_\alpha^f)^2}}\right),\end{aligned} \tag{23}$$

$$\begin{aligned}\text{Loop:}\quad & p_\alpha^i + \frac{1}{2\mu}\arctan\left(\frac{\mu\, p_\alpha^i \cos\theta_i}{\sqrt{1-\mu^2 (p_\alpha^i)^2}}\right)\\ &= p_\alpha^f - \frac{1}{2\mu}\arctan\left(\frac{\mu\, p_\alpha^f \cos\theta_f}{\sqrt{1-\mu^2 (p_\alpha^f)^2}}\right),\end{aligned} \tag{24}$$

$$\begin{aligned}\text{LUP:}\quad & p_\alpha^i + \frac{\text{arctanh}\left(\frac{\mu\, p_\alpha^i \cos\theta_i}{\sqrt{1-\mu^2 (p_\alpha^i)^2 \sin^2\theta_i}}\right)}{2\mu\sqrt{1-\mu^2 (p_\alpha^i)^2 \sin^2\theta_i}}\\ &= p_\alpha^f - \frac{\text{arctanh}\left(\frac{\mu\, p_\alpha^f \cos\theta_f}{\sqrt{1-\mu^2 (p_\alpha^i)^2 \sin^2\theta_i}}\right)}{2\mu\sqrt{1-\mu^2 (p_\alpha^i)^2 \sin^2\theta_i}}.\end{aligned} \tag{25}$$

(The remaining Algebra, the original KMM-GUP, had already been studied in depth in [14].)

These maps obviously change the reflection law and the dynamics. In particular, the two Algebras with the minus sign are substantially different: given the presence of the square root, and the fact that the value of p_α grows with time, only a finite number of reflections will be allowed before the square roots become imaginary, and therefore chaos will be removed. On the other hand, for the Brane Algebra with the plus sign, there is still an infinite number of reflections; however, chaos will also be removed. As shown in the different panels of Figure 3, for all of the three Algebras, most initial conditions will lead to oscillations between the same angles: we have found some attractors of the dynamics. This can be explained by the asymptotic behaviour of the above maps, together with the following map linking θ_f to the next θ_i:

$$\begin{cases}\theta_i' = \frac{\pi}{3} - \theta_f & \text{if} \quad \theta_f \leq \frac{\pi}{3}\,,\\ \theta_i' = -\frac{\pi}{3} + \theta_f & \text{if} \quad \theta_f > \frac{\pi}{3}\,.\end{cases} \tag{26}$$

For the Brane Algebra, given the behaviour of the reflection map for high values of p_α, the angle $\pi/6$ is the attractor. Regarding the Loop and LUP Algebras with the minus sign, we have to instead consider the asymptotic behaviour for small values of p_α (due to the square roots, it will never

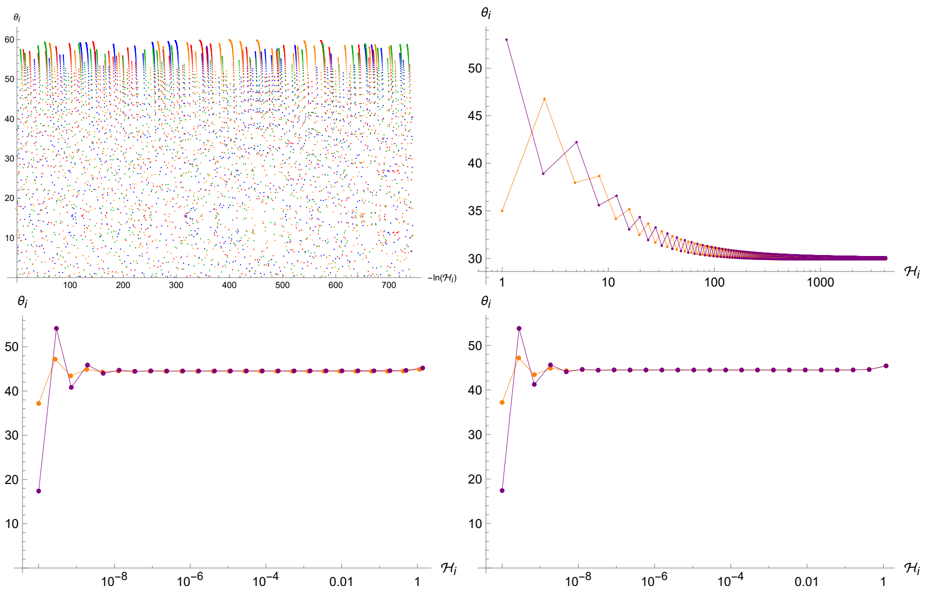

Figure 3: Bianchi IX reflection maps for four different cases, each including a few different initial conditions. Top left: classical chaotic case; top right: Brane case with attractor angle $\theta = \pi/6$; bottom: Loop (left) and LUP (right) cases, with attractor angle $\theta = \pi/4$. On the horizontal axes we have $\mathcal{H} = -p_\alpha$, coming from the reduction of the dynamics.

grow large enough), which yields the cycle $\theta_i \approx \pi/4$, $\theta_f \approx \pi/12$, $\theta_i' \approx \pi/4$, meaning that the attractor angle here is $\pi/4$. For more details, see [15].

Of course these situations should be studied more in depth through a dynamical system analysis, but we can confidently say that the elimination of Chaos is a strong consequence of this kind of Quantum Gravitational corrections, which is also consistent with what was found for the KMM Algebra in [14].

4. Conclusions

We have shown a brief overview of the formalism of Deformed Commutation Relations, and of their implementation to Primordial Cosmology. In particular, the removal of spacetime singularities is an appealing possibility in any Quantum Gravity theory, and the DCRs constitute a useful and versatile framework to implement effects from more fundamental QG theories in a simple and straightforward way. The interpretation of the Quantum Bounce as a Kasner transition in the Bianchi models is even more interesting, since no kind of matter is involved and all the degrees of freedom are purely geometrical, differently from the FLRW model that cannot sustain itself in vacuum. On the other hand, the Bianchi IX model is very relevant in modern cosmology due to its generality, and the possibility of removing chaos is appealing in respect to the problem of initial conditions; furthermore, as mentioned above, the Bianchi IX model can be used as the starting point for the general inhomogeneous cosmological solution, which is relevant for the same reasons.

It is important to stretch one last time how all these results were obtained within the classical limit of the DCRs. In isotropic cosmology it is possible to obtain a few results also on the full quantum level, but only for specific cases. Indeed, the full quantum treatment of the DCRs uncovers a few delicate issues that are still under investigation, mainly having to do with operator ordering that makes a few fundamental properties of Standard Quantum Mechanics fail. Therefore the natural next step is to tackle the quantum DCR formalism from a more formal mathematical point of view, in order to later construct full Quantum Cosmological models, or even more in general a more consistent treatment of Quantum Gravitational effects.

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